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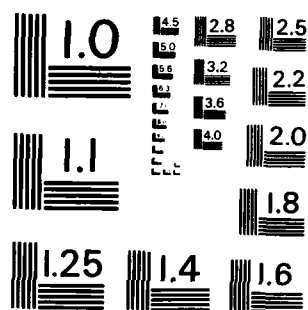
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Nonlinear Acoustics in a Dispersive Continuum:

Random Waves, Radiation Pressure,

and Quantum Noise

by

Michael Anthony Cabot

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Nonlinear Acoustics in a Dispersive Continuum:

Random Waves, Radiation Pressure,

and Quantum Noise

by

Michael Anthony Cabot

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20. ABSTRACT

The nonlinear interaction of sound with sound is studied using dispersive hydrodynamics which is derived from a variational principle and the assumption that the internal energy density depends on gradients of the mass density. The attenuation of sound due to nonlinear interaction with a background is calculated and is shown to be sensitive to both the nature of the dispersion and decay bandwidths. The theoretical results are compared to those of low temperature helium experiments. A kinetic equation which describes the nonlinear self-interaction of a background is derived. When a Debye-type cutoff is imposed, a white noise distribution is shown to be a stationary distribution of the kinetic equation. Zero point motion is introduced into the classical hydrodynamics through a renormalization scheme, which imposes the requirement that in a sound scattering the zero point motion does not lose energy. The form of the zero point motion is then determined by the kinetic equation and a derived fluid law that is analogous to Wien's displacement law for electromagnetic radiation. The kinetic equation with zero point motion included is shown to have a Planck distribution as a stationary solution. An H-theorem is presented. The attenuation and spectrum of decay of a sound wave due to nonlinear interaction with zero point motion is calculated. In one dimension, the dispersive hydrodynamic equations are used to calculate the Langevin and Rayleigh radiation pressures of wave packets and solitary waves.

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## PUBLICATIONS

M. Cabot, A. Galonsky, R. Doering, and J. Costanzo,  
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ABSTRACT OF THE DISSERTATION

Nonlinear Acoustics in a Dispersive Continuum:  
Random Waves, Radiation Pressure, and Quantum Noise

by

Michael Anthony Cabot

Doctor of Philosophy in Physics

University of California, Los Angeles, 1983

Professor Seth J. Putterman, Chair

The nonlinear interaction of sound with sound is studied using dispersive hydrodynamics which is derived from a variational principle and the assumption that the internal energy density depends on gradients of the mass density. The attenuation of sound due to nonlinear interaction with a background is calculated and is shown to be sensitive to both the nature of the dispersion and decay bandwidths. The theoretical results are compared to those of low temperature helium experiments. A kinetic equation which describes the nonlinear self-interaction of a background is derived. When a Debye-type cutoff is imposed, a white noise distribution is shown to be a stationary distribution of the kinetic equation. Zero point motion is

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## INTRODUCTION

The hydrodynamical equations for a classical one component fluid constitute a contracted description in which knowledge of a complete set of five field variables (e.g.  $\rho, s, \vec{v}$ ; the density, specific entropy, and velocity) uniquely specifies the state.<sup>1</sup> These equations are in general nonlinear due to the convective terms, which arise as a consequence of the requirements imposed by Galilean relativity, as well as the form of the equation of state for pressure "P" as a function of density which is needed to close the theory. The equations of hydrodynamics are a unique consequence of the assumption of a local description in terms of five field variables as a complete set, and Pascal's law for the isotropy of the stresses (i.e. the stress tensor in the rest frame must be diagonal).

For disturbances of sufficiently small amplitude the hydrodynamics can be linearized in the amplitude of deviation from the equilibrium state. One finds that there exist traveling wave solutions (sound waves) where the frequency  $\omega$  is related to the wave number  $k$  by

$$\omega = ck \quad \text{where} \quad c = \left( \frac{\partial P}{\partial \rho} \right)_s^{\frac{1}{2}}.$$

Such a linear relation is said to imply no dispersion. On the linear level superposition applies, sound waves run through each other without interacting so that the propagation of one wave is unaffected by the presence of other waves. Nonlinearities cause a propagating wave to interact with other waves so as to scatter energy out of the original channel and with itself so as to distort and eventually form a shock-front. The main goal of this thesis is to discuss the consequences

for sound propagation of including anisotropic stresses (and therefore dispersion) as well as nonlinear effects.

The nonlinear dispersive equations are derived from a variational principle (for an energy density which is a function of  $(\nabla \rho)^2$ ) in Chapter I, where their general properties (e.g. Kelvin, Bernoulli and conservation laws) are discussed. At lowest order the nonlinear equations take the form:

$$\square \rho = G \nabla^2 \rho^2 \quad (0-1)$$

where

$$\square \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + \alpha \nabla^4.$$

and  $G$  is a nonlinear coupling coefficient. This equation will always be solved in a perturbation approach. For instance considering a superposition of two waves which satisfy the lowest order equation  $\square \rho = 0$ ; so that

$$\rho = \rho_1 \cos(\vec{k}_1 \cdot \vec{r} - \omega_1 t) + \rho_2 \cos(\vec{k}_2 \cdot \vec{r} - \omega_2 t).$$

one finds that the leading order correction to  $\rho$  due to the interaction of the two waves is  $\rho_{12}$  where

$$\square \rho_{12} = G \rho_1 \rho_2 \nabla^2 \sum_{\pm} \cos[(\vec{k}_1 \pm \vec{k}_2) \cdot \vec{r} - (\omega_1 \pm \omega_2)t]$$

so that nonlinearities create sum and difference frequencies.<sup>2</sup>

In Appendix A we consider the energy and momentum of a 1D wave packet localized in wave number and propagating according to (0-1). As a result of the nonlinearities some of the energy and momentum are contained at (almost) zero wavelength and therefore part of the energy and momentum propagate at a different velocity than the original packet if the fluid is dispersive.<sup>3</sup> The various radiation pressures of these packets are calculated and compared with the properties of solitons.

As a propagating wave interacts with a background of sound the waves created at the sum and difference frequencies takes energy from the imposed wave and accounts for an attenuation which was calculated by Westervelt<sup>4</sup> for the nondispersive case ( $\alpha=0$ ). The main contribution to the scattering (attenuation) arises from terms on the right hand side of (0-1) which are at resonance so that

$$\omega_1 \pm \omega_2 = c^2 |\vec{k}_1 \pm \vec{k}_2|^2 + \alpha |\vec{k}_1 \pm \vec{k}_2|^4.$$

In the limit of no dispersion this occurs when two waves are parallel ( $\vec{k}_1 || \vec{k}_2$ ). The presence of a small reversible dispersion can have a dramatic effect since for  $\alpha > 0$  there is a resonance possible at some finite angle whereas for  $\alpha < 0$  no resonance is possible and the scattering of sound is dramatically reduced. In Chapter II the scattering of sound by sound for the dispersive medium is calculated. We find that Westervelt's formula is multiplied by a factor of 2 for  $\alpha > 0$  and by "0" for  $\alpha < 0$ . This effect applies, however, only when the dispersion is larger than the bandwidth which results from the scattering. In Appendix B the detailed transition formulas that apply in this intermediate regime are calculated as well as an extra (usually logarithmic)

dispersion of sound due to scattering by sound.

The results, and especially the approach employed should be of relevance to dielectric media (e.g. Helium, Quartz) at low temperature. A dielectric is a medium in which the electrons are not free to flow so that the only motion possible is a vibration of the constituent ions or molecules. These oscillations are of course sound waves and in addition to the impressed wave there will be a background of random oscillations due to the internal energy (or temperature) of the dielectric which must manifest itself in the form of sound (noise). Based upon our presumption that the nonlinear continuum mechanics can be applied to the sound wave as well as the thermal noise we can use these results of the macroscopic continuum to calculate the attenuation coefficient of sound in dielectrics. Liquid helium is a particularly rich dielectric since  $\alpha > 0$  for  $P < 18$  atmospheres and less than zero at higher pressure.<sup>5</sup> In Chapter II some of the results are compared with the measured ultrasonic attenuation properties in liquid helium.<sup>6</sup>

While nonlinear effects can cause energy to be scattered out of a given channel they can also lead to processes in the background whereby two waves collide and throw energy back into the given channel. These effects are much like restituting collisions in kinetic theory and are included in Chapter III where a Boltzmann type equation<sup>7</sup> for sound waves is derived from the hydrodynamics. The stationary solution to the nonlinear Boltzmann equation for sound is white noise (equipartition) thus deriving the noise spectrum which is normally imposed upon the linearized theory via the fluctuation dissipation theorem.



All of the results discussed above (both theory and experiment) apply to the regime where  $\hbar\omega/k_B T < 1$ . In the strong quantum limit  $\hbar\omega/k_B T > 1$  sound waves should also attenuate due to a spontaneous decay as first calculated by Slonimskii<sup>8</sup> using a quantized theory. In Chapter III we show how these results can also be obtained from the classical nonlinear hydrodynamics by including the possibility of scattering from a zero-temperature noise (or so-called zero point motion).<sup>9</sup> By the second law of thermodynamics the zero-temperature state cannot lose energy. Thus the scattering calculation must be renormalized so as not to remove energy from the zero-temperature noise. By selecting this configuration out of the classical hydrodynamics we generalize the Westervelt equation to include attenuation due to scattering by quantum as well as thermal noise. The Boltzmann equation for sound is also generalized so as to include a zero point noise defined as above. The resulting equation for the quantum theory of interacting sound waves has a stationary solution given by Planck's law plus harmonic zero point motion. Planck's constant enters as an undetermined constant which characterizes the most general solution to the balance equation. The uniqueness of this solution as well as the corresponding "H" theorem are discussed.

A central theme of this work is that many results normally thought to be obtainable only through microscopic and or quantum calculations can be interpreted and obtained directly from the classical nonlinear hydrodynamics plus the second law. Thus the approach developed here might be of value in understanding quantum effects in complex N body systems. In particular when it is desired to understand quantum

noise in macroscopic systems such as Josephson junctions<sup>10</sup> it should be realized that the usual prescription<sup>11</sup> for writing down a quantum Langevin equation does not take into account the renormalization developed in Chapter III.

As the energy density of the sound field obeys a Boltzmann type equation one can ask if there are compressional modes of the spectral intensity of sound. This, so-called second sound (thermal waves) exists in liquid helium and can be obtained via the procedure developed herein. In this vein it has also recently been shown that the complete two fluid hydrodynamics can be deduced from the nonlinear continuum mechanics of the one-component classical fluid.<sup>12</sup>

## CHAPTER I.

### THE NONLINEAR DISPERSIVE CONTINUUM MECHANICS

At low temperatures sound in liquid helium and other dielectrics decays due to scattering from other sound waves in the substance. We regard these other waves as the thermal and quantum noise present in the material. As two sound waves can interact only as a result of nonlinear processes a description of these effects must be based upon the complete nonlinear hydrodynamics. Furthermore the interaction of sound waves is strongest when the frequencies and wave numbers are in resonance or

$$\omega(k) = \omega_1 + \omega_2 \quad \text{and} \quad \vec{k} = \vec{k}_1 + \vec{k}_2, \quad (1-1)$$

where  $\omega(k)$  is the dispersion law for the medium, and we have in mind an interaction of two waves with wavenumbers  $k_1$  and  $k_2$ . Whether or not the condition (1-1) can be satisfied depends crucially upon the dispersion in the medium i.e. on the second derivative of  $\omega$  with respect to  $k$  which divides into 3 cases:

$$\frac{d^2 \omega}{dk^2} > 0 \quad \text{upward or anomalous dispersion,} \quad (1-2A)$$

$$\frac{d^2 \omega}{dk^2} < 0 \quad \text{downward or normal dispersion, and} \quad (1-2B)$$

$$\frac{d^2 \omega}{dk^2} = 0 \quad \text{nondispersive (or so-called semi-dispersion).} \quad (1-2C)$$

For cases (1-2A) and (1-2C) resonances exist whereas for case (1-2B) there is no resonant configuration for the three wave processes as described by Eq.(1-1). Thus the nonlinear effects and therefore the scattering of sound by sound is sensitive to dispersion so that a basic description of sound interacting with sound must be dispersive as well as nonlinear. We will see that for case (1-2B) the attenuation is zero to this order whereas for case (1-2A) the attenuation is twice that given by (1-2C) though there is a transition region. These three cases will be used as a guide for understanding the attenuation of sound in  $\text{He}^4$  at low temperature.

In the continuum mechanical approach dispersion arises from an internal energy which depends upon gradients of the mass density. The simplest model in which this occurs is that of particles on springs for which the Lagrangian is

$$L = \frac{1}{2} \sum_j [m\dot{\eta}_j^2 - \Phi(\eta_{j+1} - \eta_j)^2] \quad (1-3)$$

where  $\eta_j$  is the displacement of the  $j^{\text{th}}$  particle of mass  $m$  from its equilibrium position " $ja$ " where " $a$ " is the equilibrium interparticle spacing and  $\Phi$  is the spring constant. We proceed to the continuum limit by setting

$$\eta_{j+1} - \eta_j = a \frac{\partial \eta}{\partial x} + \frac{a^2}{2} \frac{\partial^2 \eta}{\partial x^2} + \frac{a^3}{6} \frac{\partial^3 \eta}{\partial x^3} + \dots$$

$$x = ja \quad m/a = \rho_e \quad (1-4)$$

$$\sum_j \rightarrow \frac{1}{a} \int dx$$

so that

$$L = \frac{\rho_e}{2} \int dx \left\{ \left( \frac{\partial \eta}{\partial t} \right)^2 - \frac{\Phi}{m} \left( a^2 \left( \frac{\partial \eta}{\partial x} \right)^2 - \frac{a^4}{12} \left( \frac{\partial^2 \eta}{\partial x^2} \right)^2 \right) \right\} \quad (1-5)$$

where  $\rho_e$  is the equilibrium mass density and we have included terms which are at most quadratic in " $\eta$ " and involve no more than four derivatives. A number of surface terms have been dropped in deriving (1-5). The equation of motion implied by (1-5) is

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} - \frac{c^2 a^2}{12} \frac{\partial^4 \eta}{\partial x^4} = 0 \quad (1-6)$$

where

$$c^2 \equiv \Phi \frac{a^2}{m}$$

is the long wavelength speed of sound. The dispersion law implied by (1-6) is

$$\omega^2 = c^2 k^2 - \frac{c^2 a^2 k^4}{12} \quad (1-7)$$

Due to the  $k^4$  term this system shows dispersion and this term arose from contributions to the potential energy proportional to  $(\partial^2 \eta / \partial x^2)^2$  which is in fact the gradient in the density, or to lowest order:

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\partial \rho}{\partial x} \frac{1}{\rho_e} \quad (1-8)$$

Using Equation (1-8) and

$$v = \frac{\partial \eta}{\partial t} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \frac{(\rho - \rho_e)}{\rho_e} ,$$

the Lagrangian (1-5) becomes to order  $(\rho - \rho_e)^2$

$$L = \int dx \frac{1}{2} \left\{ \rho_e v^2 - \frac{c^2}{\rho_e} \left( (\rho - \rho_e)^2 + \frac{a^2}{12} \left( \frac{\partial \rho}{\partial x} \right)^2 \right) \right\} \quad (1-9)$$

We see that for this simple model the potential energy terms which depend upon gradients of the density lead to the dispersion.

To obtain the general nonlinear equations including dispersion we apply Hamilton's principle to a fluid Lagrangian density given by the difference of kinetic ( $\frac{1}{2} \rho v^2$ ) and internal energy density  $U$  where

$$U = U(\rho, s, \vec{\nabla} \rho) \quad (1-10)$$

so that

$$\delta \int (\frac{1}{2} \rho v^2 - U) d\vec{r} dt = 0 \quad (1-11)$$

This, however, must be subject to the constraints of mass conservation and entropy conservation which are not contained in the symmetry of the original Lagrangian:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} = 0 \quad \text{and} \quad (1-12)$$

$$\frac{\partial \rho s}{\partial t} + \vec{\nabla} \cdot \rho s \vec{v} = 0 \quad (1-13)$$

Incorporating (1-12) and (1-13) in (1-11) with undetermined Lagrange multipliers  $\phi$ ,  $\beta$  leads to the variational principle:

$$\delta \int \left\{ \frac{1}{2} \rho v^2 - U + \phi \left( \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} \right) + \beta \left( \frac{\partial s}{\partial t} + \vec{\nabla} \cdot s \vec{v} \right) + \vec{\gamma} \cdot \left( \frac{\partial \vec{\chi}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{\chi} \right) \right\} d^3 r dt = 0 \quad (1-14)$$

where we have included also the additional constraint on the identity of fluid particles:

$$\frac{\partial \vec{\chi}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{\chi} = 0 \quad (1-15)$$

or the change in initial location  $\chi$  of a particle which is at "r" at time "t" is zero if one follows the particle's motion. This may appear tautological but is necessary to produce physical results if one performs a variation where  $\rho, v$  are fixed at the endpoints rather than the displacements.<sup>13</sup> Without (1-15) one obtains the restriction that isentropic flow must be irrotational.

Varying the independent quantities leads to

$$\begin{aligned}\delta \vec{v}: \rho \vec{v} - \rho \vec{\nabla} \phi + \beta \vec{\nabla} s + \gamma_i \vec{\nabla} \chi_i &= 0 \\ \delta \rho: \frac{1}{2} \vec{v}^2 - \frac{\partial U}{\partial \rho} + \nabla_j \frac{\partial U}{\partial \nabla_j \rho} - \frac{\partial \phi}{\partial t} - \vec{v} \cdot \vec{\nabla} \phi &= 0 \\ \delta s: \frac{\partial U}{\partial s} + \frac{\partial \beta}{\partial t} + \vec{v} \cdot \vec{\nabla} \beta &= 0 \\ \delta \chi: \frac{\partial \gamma_i}{\partial t} + \vec{v} \cdot (\vec{\nabla} \gamma_i) &= 0\end{aligned}\tag{1-16}$$

whereas varying  $\phi, \gamma, \beta$  leads to (1-12), (1-15), and (1-13). Combining equations (1-16) leads to Newton's law of motion for the dispersive continuum

$$\rho \left\{ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right\} = -\vec{\nabla} P + \frac{\partial}{\partial r_i} \left\{ \rho \vec{\nabla} \frac{\partial U}{\partial (\partial \rho / \partial r_i)} \right\} \tag{1-17}$$

where we have set

$$P = -U + \rho \mu + \rho s T + \nabla_i \rho \frac{\partial U}{\partial \nabla_i \rho}, \tag{1-18}$$

and introduced as an extension of the 1st or 2nd laws of thermodynamics:

$$dU = \mu \cdot d\rho + T d(\rho s) + \frac{\partial U}{\partial \nabla_i \rho} d(\nabla_i \rho) \tag{1-19}$$

From these basic equations (1-12, 1-13, 1-17) we can derive the laws of momentum and energy conservation

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial P_{ij}}{\partial r_j} = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + U \right) + \vec{\nabla} \cdot \vec{Q} = 0 \quad , \quad (1-20)$$

where

$$P_{ij} = P \delta_{ij} + \rho v_i v_j - \rho \frac{\partial}{\partial r_j} \cdot \frac{\partial U}{\partial (\partial \rho / \partial r_i)} \quad \text{and}$$

$$Q_i = (\mu + v^2/2 + sT) \rho v_i - \rho v_j \frac{\partial}{\partial r_i} \cdot \frac{\partial U}{\partial \nabla_j \rho} + \frac{\partial U}{\partial \nabla_i \rho} (\vec{\nabla} \cdot \rho \vec{v}) \quad .$$

Note the stress tensor is no longer diagonal so that Pascal's law is not obeyed by a dispersive system. For isentropic flows ( $s=\text{constant}$ ) Kelvin's circulation theorem and Lagrange's theorem follows as for non-dispersive fluids. For  $s$  constant (1-17) becomes:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} \left( \mu + sT - \nabla_i \cdot \frac{\partial U}{\partial \nabla_i \rho} \right)$$

so that

$$\frac{1}{2} v^2 + \mu + sT - \nabla_i \cdot \frac{\partial U}{\partial \nabla_i \rho}$$

is constant along a streamline and for irrotational flow:

$$\vec{v} = \vec{\nabla} \phi \quad \text{and} \quad \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \mu + sT - \nabla_i \cdot \frac{\partial U}{\partial \nabla_i \rho} = \text{constant} \quad (1-21)$$

throughout the fluid.



Taking the time derivative of (1-12) and subtracting the divergence of (1-20) yields

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 P = \frac{\partial^2}{\partial r_i \partial r_j} \left( \rho v_i v_j - \rho \frac{\partial}{\partial r_j} \frac{\partial U}{\partial v_i \rho} \right) \quad (1-22)$$

We will be concerned with the leading effects of dispersion so we now specialize to a system where

$$U = U_0(\rho, s) + \frac{1}{2} \Psi(\rho, s) (\vec{\nabla} \rho)^2 \quad (1-23)$$

so that 4th and higher powers of  $(\vec{\nabla} \rho)$  are ignored, in which case

$$\mu = \frac{\partial U_0}{\partial \rho} + \frac{1}{2} (\vec{\nabla} \rho)^2 \frac{\partial \Psi}{\partial \rho} \quad (1-24)$$

$$\frac{\partial U}{\partial v_i \rho} = \Psi(\rho) \nabla_i \rho \quad (1-25)$$

$$P = P_0(\rho, s) + \frac{1}{2} (\vec{\nabla} \rho)^2 \frac{\partial P_0}{\partial \rho} \quad (1-26)$$

In the perturbation method that we will employ, the fluid variables are expanded in the series:

$$\begin{aligned} \rho &= \rho_e + \delta \rho ; \delta \rho = \sum \epsilon^n \rho_n \\ \vec{v} &= \delta \vec{v} ; \delta \vec{v} = \sum \epsilon^n \vec{v}_n \\ s &= s_e + \delta s ; \delta s = \sum \epsilon^n s_n \end{aligned} \quad (1-27)$$

where  $\rho_e$  and  $s_e$  are the constant density and entropy of the equilibrium state,  $\epsilon$  is a measure of the deviation from equilibrium, and the set of functions  $(\rho_n, \vec{v}_n, s_n)$  are  $n^{\text{th}}$  order corrections to be calculated in an iterative scheme. The fluid variable expansions are substituted into the fluid equations and since the quantity  $\epsilon$  is by hypothesis small,

we demand that the equations be satisfied for each power of  $\epsilon$ . Isolating terms of the first power in  $\epsilon$ , we obtain the linearized fluid equations from which the first order solution  $(\rho_1, \vec{v}_1, s_1)$  is determined subject to given initial conditions. The  $n^{\text{th}}$  order equations are linear in the functions  $\rho_m, s_m$  and  $\vec{v}_m$ , and are comprised only of those  $\rho_m, s_m$ , and  $\vec{v}_m$  for which  $m$  is less than  $n$ . Thus, once the initial conditions are specified we can determine  $\rho, s$  and  $\vec{v}$  to any order of approximation by iterating.

The specific entropy is constant to all orders of approximation. To prove this assertion consider the  $n^{\text{th}}$  order equation for entropy:

$$\frac{\partial}{\partial \tau} s_n + \vec{v}_1 \cdot \vec{\nabla} s_{n-1} + \dots + \vec{v}_n \cdot \vec{\nabla} s_0 = 0.$$

If  $s_m$  is constant for  $m$  less than  $n$  then  $\frac{\partial}{\partial \tau} s_m = 0$  implying that  $s_m$  is constant. Since the entropy at equilibrium  $s_0$  is constant,  $s_n$  for all  $n$  is constant by induction. Thus the dependence on "s" will be suppressed from now on. Equations valid up to second order can be obtained by keeping terms of order  $(\delta\rho)^2$ . Expanding the pressure (1-26) we obtain

$$P = P_e + \left( \frac{\partial P_0}{\partial \rho} \right)_e \delta\rho + \left( \frac{\partial^2 P_0}{\partial \rho^2} \right)_e \frac{(\delta\rho)^2}{2} + \frac{1}{2} \left( \frac{\partial \rho \vec{v}}{\partial \rho} \right)_e (\vec{\nabla} \delta\rho)^2. \quad (1-28)$$

Similarly for the energy density we have

$$\rho \frac{\partial}{\partial \tau} \frac{\partial U_0}{\partial \vec{v}_i} = (\rho \vec{v})_e \left( \frac{\partial^2 \delta\rho}{\partial r_i \partial r_j} \right) + \frac{\partial \rho \vec{v}}{\partial \rho} \Big|_e \left( \frac{\partial^2 \delta\rho}{\partial r_i \partial r_j} \right) \delta\rho + \rho \frac{\partial \vec{v}}{\partial \rho} \Big|_e \left( \frac{\partial \delta\rho}{\partial r_i} \right) \frac{\partial \delta\rho}{\partial r_j} \quad (1-29)$$

so that (1-22) becomes (valid to second order in  $\delta\rho$ )

$$\square(\delta\rho) = \frac{\partial^2}{\partial r_i \partial r_j} \left\{ \left( \frac{\partial^2 P_0}{\partial \rho^2} \right)_e \frac{1}{2} (\delta\rho)^2 \delta_{ij} + \rho_e v_i v_j + \right. \\ \left. + \frac{1}{2} \left( \frac{\partial \rho \Psi}{\partial \rho} \right)_e (\vec{\nabla} \delta\rho)^2 - \delta\rho \left( \frac{\partial \rho \Psi}{\partial \rho} \right)_e \frac{\partial^2 \delta\rho}{\partial r_i \partial r_j} - \left( \rho \frac{\partial \Psi}{\partial \rho} \right)_e \frac{\partial \rho}{\partial r_i} \frac{\partial \rho}{\partial r_j} \right\} \quad (1-30)$$

where:

$$\square \equiv \frac{\partial^2}{\partial t^2} + c^2 \nabla^2 + \alpha \nabla^4, \quad \alpha \equiv (\rho \Psi)_e, \quad \text{and} \quad c^2 \equiv \left( \frac{\partial P_0}{\partial \rho} \right)_e.$$

The underlined terms in (1-30) are nonlinear as well as dispersive. These will be small for the cases which we consider as we will be concerned with the leading effects of nonlinearity and dispersion. Thus the equation which forms the main focus of this thesis is

$$\square(\delta\rho) = \frac{\partial^2}{\partial r_i \partial r_j} \left\{ \frac{(\delta\rho)^2}{2} \left( \frac{\partial^2 P_0}{\partial \rho^2} \right)_e \delta_{ij} + \rho_e v_i v_j \right\}. \quad (1-31)$$

The dispersion arises from  $\alpha$  (the dependence of  $U$  on  $(\vec{\nabla} \rho)^2$ ) and the nonlinearities are due to the doppler shift and the deviation from Hooke's law. Employing the expansion (1-27) the equation of motion for  $\rho_1$  is:

$$\square \rho_1 = 0 \quad (1-32)$$

from which  $v_1$  can be found by

$$\frac{\partial \rho_1}{\partial t} + \rho_e \vec{\nabla} \cdot \vec{v}_1 = 0. \quad (1-33)$$

The dispersion law is from (1-32):

$$\omega^2 = c^2 k^2 + \alpha k^4. \quad (1-34)$$

The equation of motion for  $\rho_2$  is

$$\square \rho_2 = \frac{\partial^2}{\partial r_i \partial r_j} \left\{ \frac{1}{2} (\rho_1)^2 \left( \frac{\partial^2 P_0}{\partial \rho^2} \right)_e \delta_{ij} + \rho_e v_{1i} v_{1j} \right\} . \quad (1-35)$$

and  $v_2$  can be determined again from the continuity law:

$$\frac{\partial \rho_2}{\partial t} + \vec{\nabla} \cdot \rho_1 \vec{v}_1 = -\rho_e \vec{\nabla} \cdot \vec{v}_2 . \quad (1-36)$$

In Appendix A we use the first and second order equations to show that a wave packet traveling in a given direction generally breaks into two pieces: one traveling at the long wavelength speed of sound and the other at the group velocity of the wave packet. We calculate the momentum, mass, and energy in these pulses and compare it to the properties of fluid solitons which are quite similar except for sign differences. The following table summarizes many of the properties of the dispersive fluid pulses and solitons.

		momentum/mechanical energy	mass	velocity
$\frac{d^2 \omega}{dk^2} > 0$	wave packet	$v_g \Lambda / (v_g^2 - c^2)$	$> 0$	$> c$
	soliton	$u \Lambda / (u^2 - c^2)$	$< 0$	$< c$
$\frac{d^2 \omega}{dk^2} < 0$	wave packet	$v_g \Lambda / (v_g^2 - c^2)$	$< 0$	$< c$
	soliton	$u \Lambda / (u^2 - c^2)$	$> 0$	$> c$

In the table,  $v_g$  and  $u$  are the group velocity and speed of the packet and soliton, and  $\Lambda = 1 + \frac{\rho}{c} \frac{dc}{d\rho}$ .

In Appendix A we also discuss the relationship of the pulse momentum to the Rayleigh and Langevin radiation pressures.

## CHAPTER II

### SCATTERING OF SOUND BY SOUND IN A DISPERSIVE CONTINUUM

In this chapter we calculate the scattering of sound by a background of sound and show how these effects are sensitive to the dispersion of the medium. We will use the dispersive fluid equations to third order in the amplitude which involves therefore two iterations of the perturbation scheme. We imagine that the background is made up of a superposition of traveling waves and focus our attention on a given impressed wave which interacts with each of the background waves via the nonlinearities. At the first order of interaction the nonlinearities generate sum and difference frequencies of the impressed wave and background. At the next order of interaction there is generated waves at the same wavenumber and frequency as the original wave. This constitutes a correction to the original wave due to its nonlinear interaction with the background and yields the attenuation.

For the impressed wave we take for the velocity and density:

$$\vec{v}_a = \hat{k}_a A \cdot \exp(i\vec{k}_a \cdot \vec{r} - i\omega_a t) + \text{c.c.} \quad \text{and} \quad \delta\rho_a = \frac{k_a}{\omega_a} \cdot \rho_e \vec{v}_a \cdot \vec{k}_a, \quad (2-1)$$

where c.c. denotes complex conjugate and  $\omega_a(k_a)$  is determined by (1-34). Similarly for the background we set

$$\vec{v}(k) = \hat{k} B(k) \cdot \exp(i\vec{k} \cdot \vec{r} - i\omega t) + \text{c.c.} \quad \text{and} \quad \delta\rho(k) = \frac{k}{\omega} \cdot \rho_e \vec{v} \cdot \vec{k}. \quad (2-2)$$

The first order equations are solved by the sum since superposition

applies in the linear approximation:

$$\vec{v}_1 = \vec{v}(\vec{k}) + \vec{v}_a \quad \text{and} \quad \rho_1 = \delta\rho(\vec{k}) + \delta\rho_a.$$

As we are interested in the change of  $\delta\rho_a$  due to the background we will neglect terms which go as  $(\delta\rho_a)^2$  which are due to the self interaction. Our assumption that the dispersion is not too strong or  $\alpha k^2/c_2 \ll 1$  causes the significant contribution to the wave scattering to arise from nearly collinear waves.

To find the first nonlinear correction  $(\rho_2, \vec{v}_2)$  we solve the second order equation which is written below with the first order solutions (2-1, 2-2) substituted:

$$\square \rho_2 = -2A\rho_e AB \sum_{\pm} (k_a \pm k)^2 \exp(iQ_{\pm}) + \text{c.c.},$$

where

$$\square \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + \alpha \nabla^4, \quad A = 1 + \frac{\rho}{c} \frac{dc}{d\rho},$$

$$\text{and } Q_{\pm} = (\vec{k}_a \pm \vec{k}) \cdot \vec{r} - (\omega_a \pm \omega)t.$$

The particular solutions for  $(\rho_2, \vec{v}_2)$  are found by trying solutions of the form:

$$\rho_2 = \sum_{\pm} V_{\pm} \exp(iQ_{\pm}) + \text{c.c.} \quad \text{and} \quad \vec{v}_2 = \sum_{\pm} \vec{W}_{\pm} \exp(iQ_{\pm}) + \text{c.c.},$$

and then solving for  $V_{\pm}$  and  $\vec{W}_{\pm}$ . The homogeneous solution, which is a solution to the linearized equations, is chosen so that the whole solution satisfies the initial conditions: the conditions being that  $\rho_n$  for  $n > 1$  vanish initially since our first order solution is constructed to give the initial density and velocity fields. The whole solution is found to be:

$$\rho_2 = \rho_e \sum_{\pm} (\vec{k}_a \pm \vec{k}) R_{\pm} + \text{c.c.} \quad \text{and} \quad \vec{v}_2 = c \sum_{\pm} (\vec{k}_a \pm \vec{k}) R_{\pm} + \text{c.c.} \quad , \quad (2-3)$$

where

$$R_{\pm} = -2\Lambda AB(\vec{k}_a \pm \vec{k}) \exp(i(\vec{k}_a \pm \vec{k}) \cdot \vec{r}) \frac{(\exp(-i(\omega_a \pm \omega)t) - \exp(-i\omega_{\pm}t))}{\omega_{\pm}^2 - (\omega_a \pm \omega)^2}$$

and

$$\omega_{\pm} = \frac{(\vec{k}_a \pm \vec{k})}{|\vec{k}_a \pm \vec{k}|} \omega(\vec{k}_a \pm \vec{k}) \quad .$$

That the sound wave amplitude must change in the process of producing  $(\rho_2, \vec{v}_2)$  can be proven using energy conservation. We find the quantitative change from  $\rho_3$  which is determined by solving the third order, perturbation equation:

$$\square \rho_3 = \frac{\partial^2}{\partial r_i \partial r_j} \left\{ \left( \frac{\partial^2 p}{\partial \rho^2} \right)_e \rho_1 \rho_2 \delta_{ij} + \rho_e (v_{1i} v_{2j} + v_{2i} v_{1j}) \right\} \quad (2-4)$$

where we have ignored terms cubic in first order quantities since their effect, which is studied in Appendix B, is to alter the sound wave speed but not the amplitude. Also in Appendix B we show that there are dispersion effects that have a logarithmic dependence on the sound frequency. The attenuation effects are stronger since they result from resonant mode coupling at both orders of iteration.

The change in the sound wave amplitude is identified as the amplitude of that harmonic component of  $\rho_3$  which has the same wave-number, frequency, and phase as the sound wave. This harmonic component is generated only by those terms in the differential equation corresponding to these same characteristics. Therefore, when we write the



equation (2-4) with first and second order solutions (2-1, 2-2, 2-3) substituted, we remove all but the relevant terms:

$$\square \rho_3 = -A|B|^2 |4\rho_e c k_a^2 A^2 \exp(i\vec{k}_a \cdot \vec{r} - i\omega_a t) \cdot \int_{\pm} ((\omega_{\pm} \mp \omega)^2 - \omega_a^2) Q_{\pm} + c.c. ,$$

where

$$Q_{\pm} = (k_a \pm k)^2 \frac{\cos((\omega_{\pm} - \omega_a \mp \omega)t) - 1}{((\omega_{\pm} \mp \omega)^2 - \omega_a^2) \cdot (\omega_{\pm}^2 - (\omega_a \pm \omega)^2)} .$$

The relevant solution, which is found in the same way as for  $\rho_2$ , is

$$\rho_3 = A|B|^2 |4\rho_e c k_a^2 A^2 \exp(i\vec{k}_a \cdot \vec{r} - i\omega_a t) \int_{\pm} Q_{\pm} + c.c. .$$

The change in the sound wave amplitude is the amplitude of this solution:

$$\Delta A = 4A|B|^2 |c^2 A^2 k_a^2 [ Q_- + Q_+ ] . \quad (2-5)$$

At resonance where  $\omega_{\pm} - (\omega_a \pm \omega) = 0$ , the  $Q_{\pm}$  grows quadratically in time whereas off resonance the  $Q_{\pm}$  is bounded and oscillatory. When dispersion is weak ( $\frac{\alpha k^2}{c^2} \ll 1$ ), resonance occurs when the angle between the directions of propagation of the sound and background waves satisfies

$$\theta^2 \approx \frac{3\alpha}{c^2} (k_a + k)^2 \quad \text{or} \quad (\theta - \pi)^2 \approx \frac{3\alpha}{c^2} (k_a - k)^2 ,$$

from which we infer that for resonance the waves must be nearly collinear and  $\alpha$  must be nonnegative.

To find the change in the sound wave amplitude due to nonlinear interaction with a background distribution, we integrate the change due to the single background wave treated as a differential piece. To

characterize the background, we let  $E(k, \hat{n})$  be the background energy density per unit wavenumber at  $k$  and per unit solid angle in the  $\hat{n}$  direction. We choose the amplitude  $B$  of the single background wave so that its wave energy equals the background energy in the interval  $dk$  at  $k$  and in the solid angle  $d\Omega$  at  $\hat{k}$ . Since the wave energy density to second order is  $\frac{1}{2} \rho_e |B|^2$ , we choose  $B$  to be given by

$$|B|^2 = \frac{2}{\rho_e} E(k, \hat{k}) \cdot dk \cdot d\Omega$$

Using (2-5), we integrate the effect of this differential piece to find the change in the sound wave amplitude due to the background distribution:

$$\Delta A = A \frac{c^2}{\rho_e} A^2 k_a^2 \int dk \int_{\pm} I_{Q\pm}$$

where 
$$I_{Q\pm} = \int d\Omega E(k, \hat{k}) \cdot Q_{\pm}$$

The angular integration can be approximated for long times because the quantity  $Q_{\pm}$  becomes peaked about resonance. Since the main contribution to the integral comes from a solid angle centered at resonance and whose width is inversely proportional to time, we can approximate the integral by expanding the integrand in the angle about resonance:

$$I_{Q\pm} \approx - \int_0^{\pi} d\theta \cdot 2\pi\theta \cdot \frac{E(k, \hat{k})}{4k_a c^2 (k_a \pm k)} \cdot \frac{\sin^2((-P_{\pm}\theta^2/2 + R_{\pm})t/2)}{(-P_{\pm}\theta^2/2 - R_{\pm})}$$

where

$$P_{\pm} = \frac{ckk_a}{|k_a \pm k|} \quad \text{and} \quad R_{\pm} = \frac{3\alpha}{2c} \cdot k |k_a \pm k|$$

Using the substitution  $y = -\frac{1}{2} P \pm \theta^2 + R \pm$  and for large times letting the upper limit of integration extend to infinity, we are left with

$$I_{D\pm} = -t \cdot \frac{\pi^2}{8P\pm} \cdot E(k, k_a) \cdot I \quad (2-6)$$

where

$$I = \frac{2}{\pi} \int_{-R_{\pm}t/2}^{\infty} \frac{\sin^2(y)}{y^2} dy$$

For times such that  $|tR_{\pm}| \gg \pi$ , the value of  $I$  depends on the sign of  $\alpha$ :

if  $\alpha < 0$ , then  $I \approx 0$ ,

if  $\alpha > 0$ , then  $I \approx 2$ .

When  $|tR_{\pm}| \ll \pi$  or if  $\alpha=0$ , then  $I=1$ .

That Westervelt's prediction for the attenuation of sound in liquid helium is for some cases roughly half the experimental value is simply explained by the sensitivity of the attenuation upon the nature of the dispersion.<sup>14</sup> Using the result (2-6), the change in the sound wave amplitude due to the background is  $\Delta A = -tA\beta$ , where  $\beta$ , which is defined as the attenuation coefficient, is given by

$$\beta(k_a) = \frac{\pi^2}{2\rho_e c} A^2 I \left\{ \int_0^{\infty} \frac{(k_a + k)^2}{k} E(k, k_a) dk - \int_{k_a}^{\infty} \frac{(k_a - k)^2}{k} E(k, k_a) dk + \int_0^{k_a} \frac{(k_a - k)^2}{k} E(k, k_a) dk \right\} \quad (2-7)$$

Treating thermal motion as the background, Westervelt used nondispersive hydrodynamics for his prediction for which  $\alpha$  is zero and the value of  $I$  is unity.<sup>4</sup> However if  $\alpha > 0$ , then the value of  $I$  can double.

The attenuation coefficient (2-7) has three terms each of which

corresponds to a specific process. The first term is the attenuation due to those nonlinear interactions for which the sound wave and the background waves interact to produce waves at the sum wavenumber  $(\vec{k} + \vec{k}_a)$ . The second and third terms correspond to the processes that produce waves at the difference wavenumber  $(\vec{k}_a - \vec{k})$ . The second term, which is the contribution from those background waves for which  $k_a > k$ , is negative and therefore the sound wave is amplified in these processes.

Media for which  $\alpha > 0$  have a dispersive nature that has been labeled anomalous. Our experience with most materials is that  $\frac{d^2\omega}{dk^2} \leq 0$ , however upon examination of the dispersion relation (1-34) it is evident that a positive  $\alpha$  implies that  $\frac{d^2\omega}{dk^2} > 0$ .

The specific heat measurements of Greywall<sup>5</sup> indicate that the dispersion of liquid helium at low pressures is anomalous for long wavelengths.

With the implications of the specific heat measurements at high pressures,<sup>5</sup> we can explain by using our calculations why the attenuation of sound in liquid helium decreases dramatically when the pressure is raised past 18 atmospheres. The specific heat measurements show that the dispersion of liquid helium becomes normal ( $\alpha < 0$ ) at pressures above 18 atmospheres. However if  $\alpha$  changes from positive to negative, our calculations have shown that the value of  $I$ , upon which the predicted attenuation is proportional, drops approximately from two to zero.

For long times the decay of the sound wave in the presence of a background is exponential. Though in our perturbation scheme we implicitly assumed that the sound amplitude was nearly constant, we extend its validity by taking the amplitude to be time dependent. Then

in the interval  $(t, t+dt)$  the change in the amplitude is given by the differential equation:

$$dA = -\beta \cdot A(t) dt ,$$

which is integrated to give  $A(t) = A(0) e^{-\beta t}$ .

The attenuation of sound in liquid helium at low temperatures is understood as resulting from the nonlinear interaction of the sound with the background of thermal motion. It is known from statistical mechanics that the thermal background is isotropic with the energy density per unit wavenumber given by the Planck distribution.

To obtain a numerical value for the attenuation of sound in liquid helium at low temperatures, we will insert into our calculated attenuation the values for the speed of sound ( $c$ ), the density ( $\rho$ ) and the Gruneisan constant ( $\Lambda$ ) which are given by the experiments of Abraham et al.<sup>15</sup> The predicted attenuation will then be compared to the experimental results<sup>6</sup> for which the attenuation was measured for sound frequencies between 10 and 260 megahertz, temperatures between .1° and 1°K, and pressures between 0 and 25 atmospheres.

For the range of frequencies and temperatures covered by the experiments, the expression for the calculated attenuation takes the following approximate form:

$$\beta = \left( \frac{2\pi^6 \Lambda^2 k_b^2}{15\rho_e h^3 c^6} \right) I \omega_a^4 T^4 . \quad (2-8)$$

where  $k_b$  is Boltzmann's constant and  $h$  is Planck's constant. The simplification arises because in the experiments the sound frequency and the temperature satisfy the condition:  $\frac{h\omega}{k_b T} \ll 1$ . This condition

implies that most of the energy in the Planck distribution resides in wavenumbers much greater than the sound wavenumber, which allows us to take as zero the value of the sound wavenumber as it appears in the limits of integration in the attenuation expression (2-7). The attenuation then becomes proportional to the total energy which is given by statistical mechanics.

Using the experimentally determined values of the bracketed quantities in (2-8), we find the predicted attenuation coefficient for sound in liquid helium at zero pressure as a function of frequency (in radians per second) and temperature (in degrees Kelvin) to be

$$\beta = 1.5 \times 10^{-6} I \omega_a T^4 \text{ (dB/cm) .}$$

Figure 1 is the plot of the predicted (I=1 and I=2) and measured attenuations versus temperature for a 15 megahertz sound wave in liquid helium at zero atmospheres. (The figures are on pages 29-32.)

Figure 2 is a plot of the measured attenuation versus frequency at zero atmospheres and .2° Kelvin. The solid lines are the predictions of our calculation (I=2) and Westervelt's (I=1).

Figure 3 displays the measured and predicted (I=1 and I=2) attenuations versus pressure for a 256 megahertz sound wave in helium at .22° Kelvin.

Figure 4 shows the measured attenuation versus temperature for a 15 megahertz sound wave at a pressure of 25 atmospheres. The predicted attenuation is zero because the dispersion is normal ( $\alpha < 0$ ) above 18 atmospheres. For temperatures above .6° Kelvin, the attenuation is not zero as predicted because the sound is interacting with "rotons" which

are high frequency thermal components whose energy grows rapidly with temperature.

We can explain by considering bandwidth effects why the experimental points in Figures 1 and 2 might lie between our prediction ( $I=2$ ) and Westervelt's ( $I=1$ ). Bandwidth effects result from the uncertainty in frequency inherent in any decaying waves. In our case both the sound and thermal waves are decaying and if these decays are large then the approximations made in evaluating the integrals over the background distribution become poor. In Appendix B, we use a Green's function technique to show that the effect of this bandwidth is to multiply our calculated attenuation by the factor:

$$\frac{1}{2} \left\{ 1 + \frac{2}{\pi} \arctan \left( \frac{3}{2} \alpha \left( \frac{2\pi k_b T}{hc} \right)^2 \omega_a \tau \right) \right\} ,$$

where  $\tau$  is the lifetime of a typical background wave. Including this factor, we find our attenuation is equal to Westervelt's at frequencies where  $\alpha \left( \frac{k_b T}{hc} \right) \omega_a \tau \ll 1$  and is twice Westervelt's at sound frequencies where the inequality is reversed.

The frequency dependence of the attenuation predicted by using the bandwidth effect does not agree with the experimental points of Figure 2. Possible explanations for the discrepancy between theory and experiment may arise from considering the following: misinterpretation of the data, long thermal lifetimes, or attenuation due to interaction with second sound. For the first point, the experimental data may have systematic error because to find absolute attenuations, risky assumptions were made about the temperature dependence of the attenuation.<sup>6</sup>

Secondly, it may be incorrect to assume as we did in calculating

the bandwidth effect that the thermal lifetime is shorter than the sound wave lifetime. Although the dispersion in liquid helium is anomalous at low wavenumbers, it is always normal at sufficiently high  $k$ . Thus there is an inflection point  $\left(\frac{d^2\omega}{dk^2}=0\right)$  at some finite  $k=k_1$ . If the background is concentrated at wavenumbers greater than  $k_1$ , then the background would be quite long lived, perhaps with a lifetime longer than the impressed sound wave. This case would correspond to a dispersion relation such as  $\omega^2=c^2k^2+\alpha k^4-\alpha'k^6$ . Although the results will be different from those calculated here we believe that the same procedure will work and may account for the discrepancy.

Finally there can appear in the distribution of sound waves compressional modes of the local energy density or so-called second sound. The appearance of the new mode is a spontaneous symmetry breaking in the nonlinear hydrodynamics. Now there can appear new nonlinear processes in which the sound wave interacts with second sound and that can still further complicate the comparison of these results with the properties of liquid helium.<sup>16</sup>



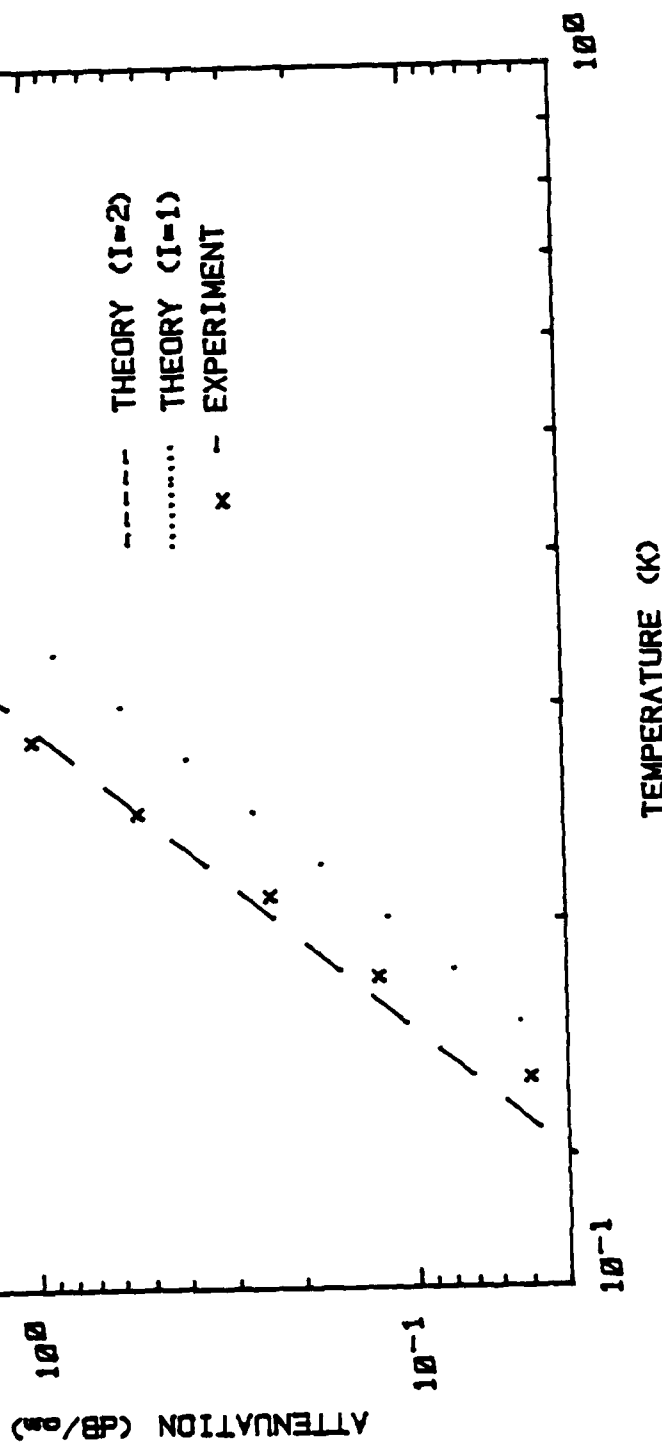


FIGURE 1. Attenuation vs. temperature for a 15 MHz sound wave at zero pressure.

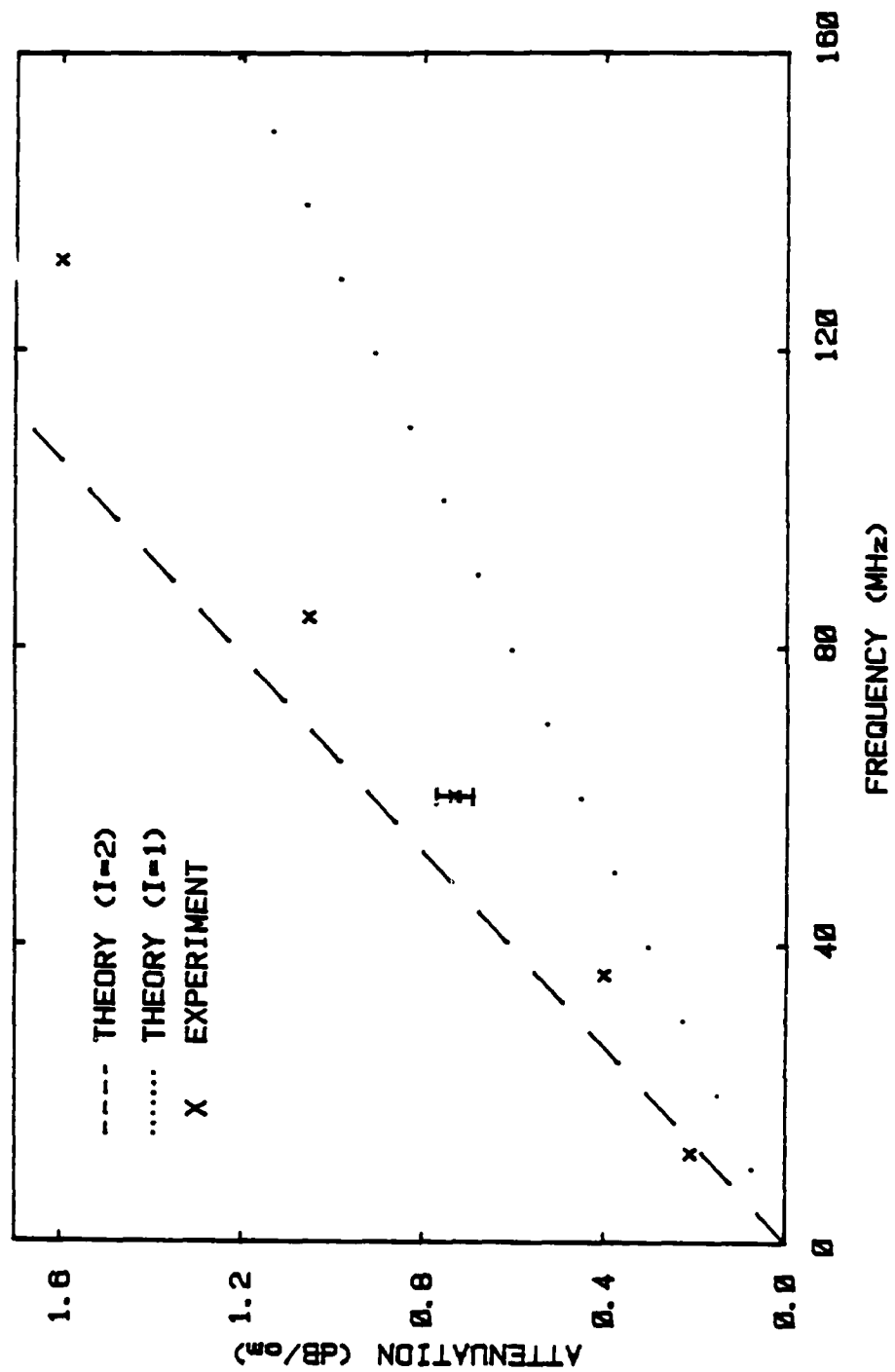


FIGURE 2. Attenuation vs. frequency at zero pressure and a temperature of 0.20 K.

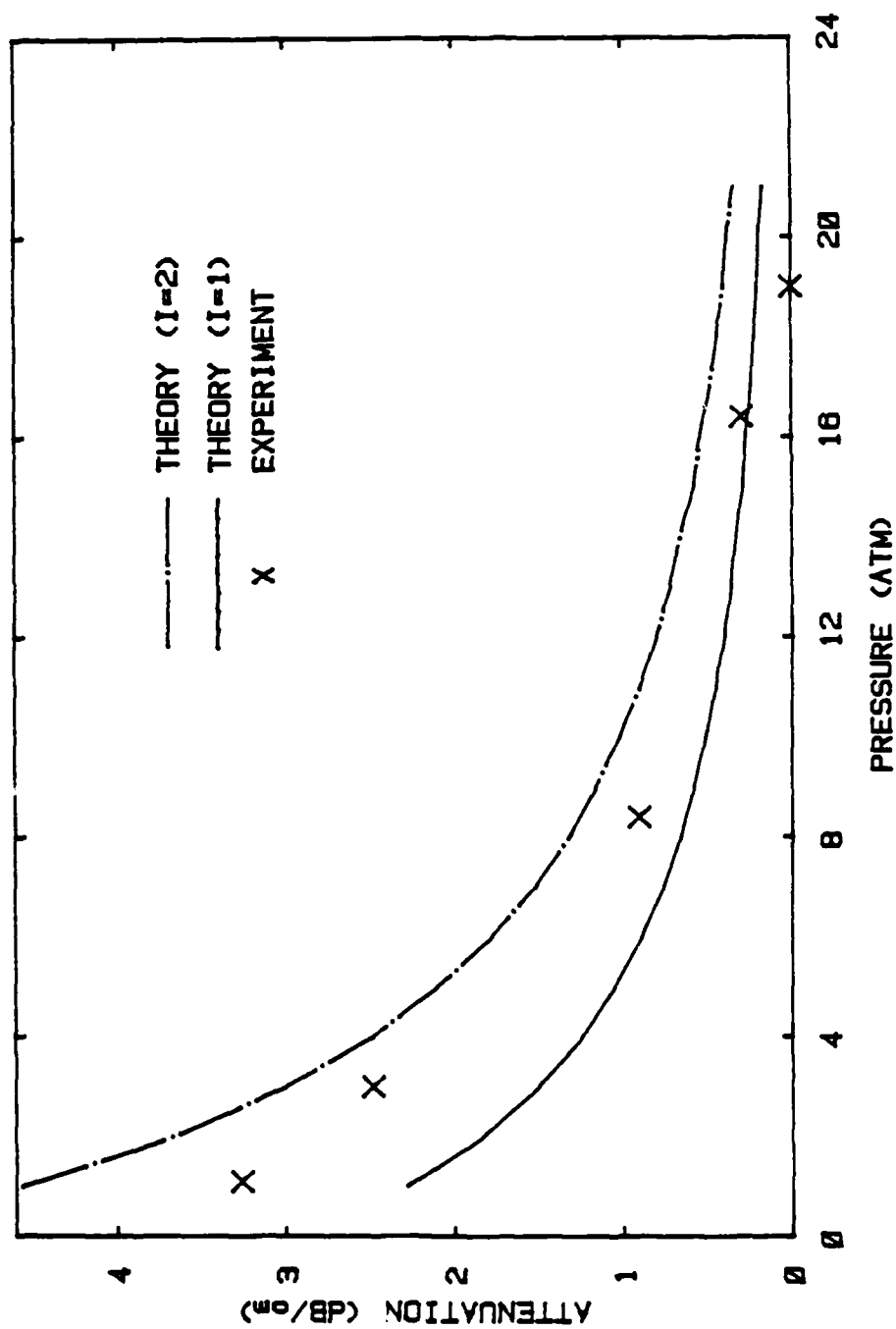


FIGURE 3. Attenuation vs. pressure for a 258 MHz sound wave at a temperature of 0.22 K.

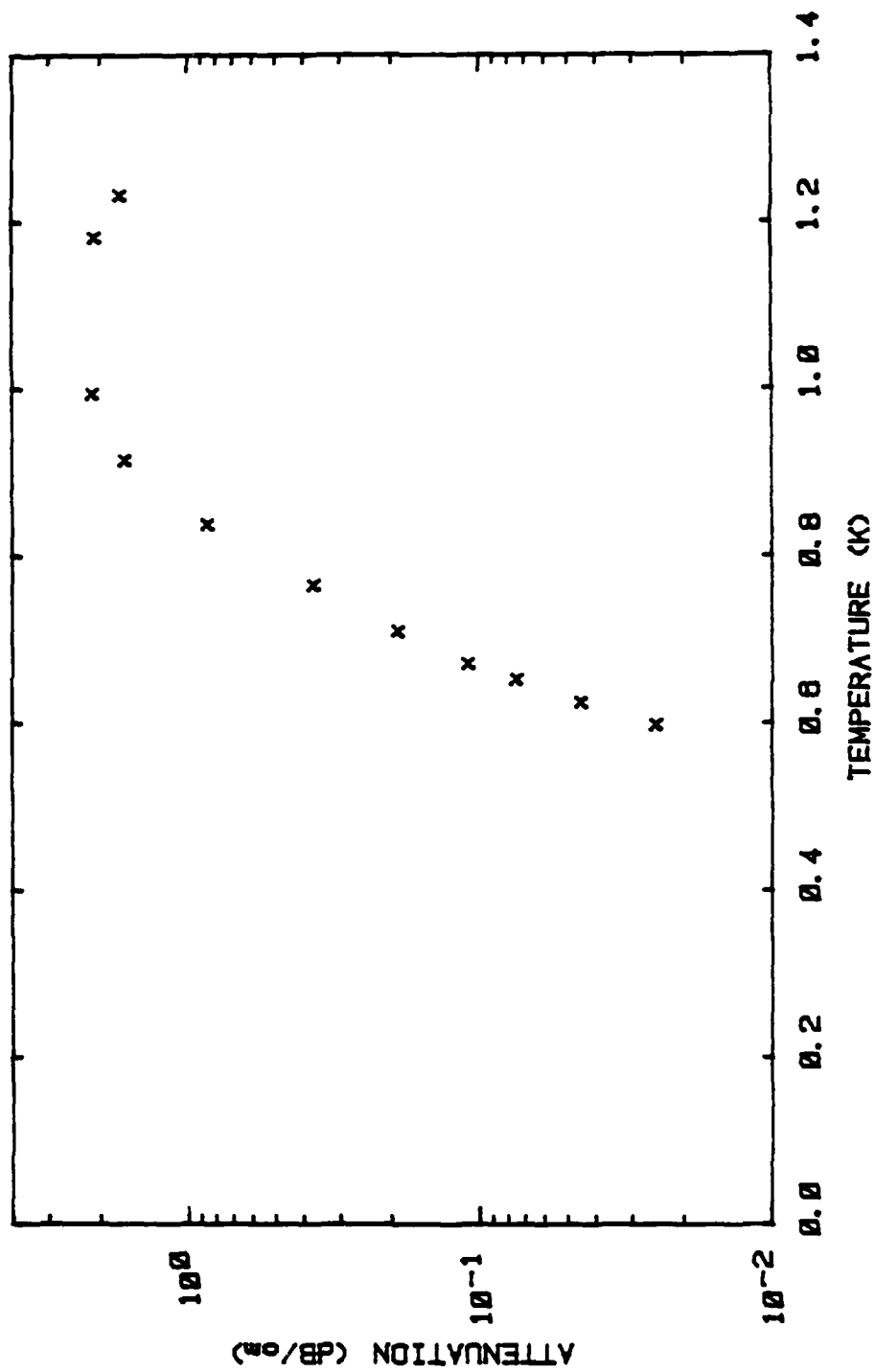


FIGURE 4. Attenuation vs. temperature of a 15 MHz sound wave at pressure of 25 Atm.

### CHAPTER III

#### BOLTZMANN EQUATION FOR SOUND

In the last chapter, where we considered the nonlinear interaction between a sound wave and a background, we assumed that the background energy distribution was constant in time. In this chapter, we determine the nonlinear self-interaction of an isotropic background, and express how the background distribution changes in time via a kinetic equation derived from the hydrodynamics. If we impose a Debye type cutoff, the stationary background distribution of the kinetic equation is a "white noise" distribution. Next we present a reasonable method of adding the zero point motion contributions to the kinetic equation. The form of the zero point distribution is determined using the kinetic equation and a law analogous to Wien's law for radiation. With this form, the improved kinetic equation has a stationary solution that is a Planck distribution. The chapter ends with the presentation of a classical H-theorem and a discussion of some effects of zero point motion.

To simplify the discussion we will impose the restrictions that the medium be nondispersive and that the background be low enough in energy so that its self-interaction can be considered as being the collection of binary interactions between the harmonic components that constitute the background.

Our immediate goal is to derive a kinetic equation which gives the time rate of change of  $u(k)$ , the background energy per unit

wavenumber at  $k$  or so-called spectral intensity. This background energy at  $k$  changes not only by its direct interaction with the rest of the background but also by restituting processes resulting from the binary interaction of other background pieces. The change due to direct interaction is determined directly from the attenuation of sound calculated in the last chapter. The effect of the restituting processes is found with the additional use of energy conservation which is a property of the nonlinear continuum mechanics.

For an isotropic background, the time rate of change of the background energy at  $k$  due to direct interaction is the same as that of a sound wave of wavenumber  $k$  and equal energy. Having shown that for a sound wave the rate of change of the density amplitude,  $A$ , is given by

$$\frac{d}{dt} A(k_a) = -\beta(k_a)A(k_a) \quad ,$$

and knowing that energy is quadratic in amplitude, we deduce that the time rate of change of the background energy density  $u(k)$  for direct interaction is expressed by

$$\frac{d}{dt} u(k) = -2\beta(k)u(k) \quad ,$$

where  $\beta(k)$  in terms of  $u(k)$  was derived in the last chapter:

$$\beta(k) = \frac{\pi}{8\rho_e c} A^2 \left\{ \int_0^{\frac{(k+q)^2}{q}} u(q) dq + \int_0^{\frac{k(k-q)^2}{q}} u(q) dq - \int_k^{\frac{(k-q)^2}{q}} u(q) dq \right\}.$$

To complete the kinetic equation, we must find the rate of change of the background energy density due to restitution. When two waves interact they produce waves whose wavenumbers are equal to the sum and difference of the wavenumbers of the interacting waves. It follows that in a restituting process, the background energy at  $k$  changes

because the wavenumber  $k$  is either the sum or difference of the wavenumbers of many pairs of directly interacting background pieces. The changes due to the sum and difference cases will be treated independently and will be calculated for single pairs of interacting background pieces by using energy conservation from which we know that the energy gained in restitution is the energy lost by direct interaction. The total restitution will be given by an integral over all possible interacting pairs.

We begin by focusing on the single sum process in which the background energy in the interval  $\Delta K$  at wavenumber  $(k-q)$  interacts with the energy in the interval  $dq$  at  $q$  to reconstitute energy to the background at the sum wavenumber  $k$ . The time rate of energy loss of these interacting waves can be obtained by dissecting our kinetic equation for direct interaction, which is

$$\frac{d}{dt} u(k) = -2\beta(k)u(k) \quad . \quad (3-1)$$

The attenuation coefficient for a sound wave of wavenumber  $k$  resulting from just sum interaction with a background is given by the first term in the brackets of the expression for  $\beta(K)$ :

$$\beta_{\text{sum}}(K) = \frac{Q}{2} \int_0^K \frac{(K+q)^2}{q} u(q) dq \quad , \quad \text{where} \quad Q = \frac{\pi}{4\rho_e c} \Lambda^2 \quad .$$

Singling out the contribution of the energy in the interval  $dq$ , substituting into Eq. (3-1) and letting  $K=k-q$ , we find that the time rate of change of the background energy in  $\Delta K$  at  $(k-q)$  for our single sum process is

$$\frac{d}{dt} u(k-q)\Delta K = -Q \frac{k}{q} u(q) dq u(k-q)\Delta K \quad .$$

Likewise, the rate of change of the energy in  $dq$  at  $q$  is

$$\frac{d}{dt} u(q) dq = -Q \frac{k^2}{(k-q)} u(k-q) \Delta K u(q) dq .$$

Since the energy lost by both waves in direct sum interaction is gained in restitution in the interval  $\Delta K$  at  $k$ , we deduce that

$$\frac{d}{dt} u(k) = Q \frac{k^2}{q(k-q)} u(k-q) u(q) dq .$$

Integrating this expression over all possible pairs of interacting waves, we have the rate of change of the background energy at  $k$  caused by the restituting sum interactions:

$$\frac{d}{dt} u(k) = Q \int_0^{k/2} \frac{k^3}{q(k-q)} u(k-q) u(q) dq . \quad (3-2)$$

The contribution to the kinetic equation from the restituting difference processes, which is calculated in the same manner as the sum processes, is

$$\frac{d}{dt} u(k) = Q \int_0^{\infty} \frac{k^3}{q(k+q)} u(k+q) u(q) dq . \quad (3-3)$$

In calculating this contribution, we use the attenuation coefficient for sound interacting with a background via difference processes:

$$\beta_{dif}(k) = \frac{Q}{2} \left\{ \int_0^k \frac{(k-q)^2}{q} u(q) dq - \int_k^{\infty} \frac{(k-q)^2}{q} u(q) dq \right\} . \quad (3-4)$$

The significance of the two terms is that when two waves interact in a difference process, the wave of lesser wavenumber gains energy while the wave of greater wavenumber loses.

The complete kinetic equation obtained by adding the contributions of the direct process (3-1) to that of the sum and difference



restituting processes (3-2, 3-3) is

$$\frac{1}{Q} \frac{d}{dt} u(k) = \left\{ \frac{-\beta(k)u(k)}{Q} + \int_0^{k/2} \frac{k^3}{q(k-q)} u(k-q)u(q) dq + \right. \\ \left. + \int_0^{\infty} \frac{k^3}{q(k+q)} u(k+q)u(q) dq \right\} \quad (3-5)$$

We will refrain from application of this equation until we improve it with the inclusion of zero point motion contributions. However, we mention that if a wavenumber cutoff is imposed analogous to that which Debye argued for solids, then the kinetic equation has a stationary distribution (i.e.,  $\frac{du(k)}{dt} = 0$ ) of the form:

$$u(k) = \text{constant} \times k^2,$$

which is representative of the statistical mechanic notion of equipartition of energy.

Our attention is now directed toward determining how the kinetic equation is altered by zero point motion. Though the effects of zero point motion has traditionally been thought to be outside the domain of classical physics, we will show that it is possible to introduce zero point motion in a classical vein. The zero point motion is postulated as an immutable energy distributed uniquely over all wavenumbers, existing even in the absence of thermal motion, but nevertheless able to interact in some restricted sense with other waves.

The implementation of our postulate becomes apparent when we consider a sound wave in the presence of a zero point wave. We would like to treat the interaction of the sound wave with the zero point wave in

the same way as any two waves, but since the energy in a channel which has only zero point energy cannot diminish, we must somehow restrict the interaction. For the case of two waves interacting in a sum process, both waves lose energy; therefore if one of the waves is a zero point wave whose energy cannot diminish, the sum process must be eliminated. In the difference process, the wave of lesser wavenumber gains energy and the wave of greater wavenumber loses. As in the sum process, the possibility that a wave of greater wavenumber in the difference process be a zero point wave must be eliminated, but we may allow a sound wave to interact with a zero point wave of lesser wavenumber if the energy normally gained at the zero point wavenumber is placed into sound wave energy at that wavenumber.

To calculate the zero point contributions to the kinetic equation, we utilize that the interactions of the background waves with the zero point waves is just a subset of the interactions of the background self-interaction whose contributions to the kinetic equation have been calculated. The time rate of change of the background energy at  $k$  caused by zero point motion is due to difference processes of three types:

1. The direct interaction of the background energy at  $k$  with zero point waves of wavenumber less than  $k$ ,
2. The direct interaction of the zero point energy at  $k$  with background waves of wavenumber greater than  $k$ ,
3. Restitution from the background energy at  $(k+q)$  interacting with the zero point energy at  $q$  (for all  $q$ ).

For each of these types, the time rate of change of the background

energy at  $k$  caused by the background has been calculated to be, respectively,

$$(1) \quad -u(k) \int_0^k \frac{(k-q)^2}{q} u(q) dq ,$$

$$(2) \quad u(k) \int_k^\infty \frac{(k-q)^2}{q} u(q) dq ,$$

$$(3) \quad \int_0^\infty \frac{k^3}{q(k+q)} u(k+q)u(q) dq .$$

As an example of how this procedure works consider the change in energy in channel  $k$  due to sound in channel  $q < k$  due to difference processes:

$$\frac{d}{dt} u(k) dk = -Q \frac{(k-q)^2}{q} u(q) dq u(k) dk , \quad (3-6)$$

where we have used (3-1, 3-4). Similarly for  $q > k$ , we have

$$\frac{d}{dt} u(q) dq = Q \frac{(k-q)^2}{k} u(k) dk u(q) dq . \quad (3-7)$$

Next on the right hand sides of these expressions we replace  $u$  with  $u+v$  and focus attention on the contribution linear in the zero point energy per unit wavenumber, which we will label  $v$ , (the terms quadratic in  $u$  have already been incorporated into (3-5)):

$$\frac{d}{dt} u(k) = -Q \frac{(k-q)^2}{q} [ u(q)v(k) + u(k)v(q) ] dq \quad (3-8a)$$

$$\frac{d}{dt} u(q) = Q \frac{(k-q)^2}{k} [ u(q)v(k) + u(k)v(q) ] dk \quad (3-8b)$$

so that from (3-8a) we see that the process whereby a zero point fluctuation of wavenumber  $k$  interacts with an excitation of wavenumber  $q$  is forbidden since it removes energy from channel  $k$ . Deleting this term from (3-8a) and (3-8b) yields direct and restituting contributions to the change in energy density given by expressions (3-6) and (3-7) above with  $u(q)$  replaced with  $v(q)$ . Replacing also the background energy density of lesser wavenumber in the  $\beta(k)$  in (3-1) with the zero point energy density (i.e.  $u(k) \rightarrow v(k)$ ) we find the zero point contribution to the kinetic equation which combined with (3-5) yields the complete kinetic equation:<sup>17</sup>

$$\begin{aligned} \frac{1}{Q} \frac{d}{dt} u(k) = & \left\{ -u(k) \left( \frac{\beta(k)}{Q} + \int_0^k \frac{(k-q)^2}{q} v(q) dq \right) + \right. \\ & \int_0^{k/2} \frac{k^3}{q(k-q)} u(k-q) u(q) dq + \int_0^{\infty} \frac{k^3}{q(k+q)} u(q) u(k+q) dq \\ & \left. + v(k) \int_k^{\infty} \frac{(k-q)^2}{q} u(q) dq + \int_0^{\infty} \frac{k^3}{q(k+q)} v(q) u(k+q) dq \right\}. \end{aligned}$$

The stationary solution of the kinetic equation is the background equilibrium distribution. To find this stationary solution, we must first determine the form of the zero point distribution.

Using the kinetic equation and a fluid law analogous to Wien's law for radiation, we are able to show using the second law of thermodynamics that the zero point distribution is of the form:

$$v(k) = A c k^3,$$

where  $A$  is an undetermined constant. The fluid law, which is proven in Appendix C, states that the background equilibrium distribution must have the form:

$$u(k) = c k^3 f(ck/T),$$

where the function  $f$  is as yet unknown. Substituting this form into the kinetic equation we find that all the terms involving quadratic powers of  $u$  yield expressions which are homogeneous functions of  $T$  and  $k$  of the eighth degree. That is replacing  $T$  and  $k$  with  $\alpha T$  and  $\alpha k$  yields  $\alpha^8$  times the original expression. Thus if there is to be a non-trivial balance between the zero point terms and the thermal terms the expressions involving  $v$  must also be homogeneous of degree 8 which means that  $v$  is proportional to  $k^3$ .

The equilibrium background distribution, which is the stationary solution to the kinetic equation subject to Wien's law, is a Planck distribution:

$$u(k) = 2A\alpha k^3 (\exp(\gamma ck/T) - 1)^{-1} ,$$

where  $\gamma$  is a second undetermined constant. That this is a stationary solution to the kinetic equation can be shown directly with the following identities:

$$f(q)f(k+q) = f(k)f(q) \pm f(k)f(k\mp q) \pm f(k+(q\pm q)/2) ,$$

where  $f(k) = (\exp(\alpha k) - 1)^{-1}$ . Of the two constants,  $A$  and  $\gamma$ , only one must be found by experiment since at high temperatures the equipartition of energy determines the ratio:

$$\frac{A}{\gamma} = \frac{k_b}{2\pi^2} .$$

Einstein and Stern<sup>18</sup> by considering the scattering of a charged particle by an electromagnetic field attempted to derive the Planck distribution from classical mechanics by assuming a particular zero point noise spectrum. Their results were off by factors of two. Here by considering self-consistently the scattering of sound by sound we have arrived at the proper thermal distribution as well as the zero point spectral intensity. These arguments have now been extended to the nonlinear electromagnetic equations by Roberts.<sup>19</sup>

That an "H-theorem" exists can be proven with the kinetic equation. If we let H be defined as

$$H = \int_0^\infty d(\ln k) [ (u(k)+2v(k)) \cdot \ln(u(k)+2v(k)) - u(k) \cdot \ln(u(k)) ] ,$$

then using the kinetic equation and the cubic form of the zero point motion we can show that

$$\frac{dH}{dt} = \frac{1}{2} Q^{-1} \int_0^\infty dK \int_0^K dq \frac{u(q)u(K)u(K-q)}{A^2 qK(K-q)} (a-b) \ln(a/b) ,$$

where

$$a = \left(1 + \frac{q^3}{u(q)}\right) \cdot \left(1 + \frac{(K-q)^3}{u(K-q)}\right) \quad \text{and} \quad b = (1 + K^3/u(K)) .$$

Since the integrand is always positive, we conclude that  $\frac{dH}{dt} \geq 0$ , which is the H theorem. Demanding that H is stationary and imposing Wien's law for fluids, we again obtain the background equilibrium distribution.

As a final application of the kinetic equation, we calculate the attenuation and decay spectrum of a sound wave interacting with zero point motion. The sound wave is represented in terms of  $u(k)$  as follows:

$$u(q) = u_0 \delta(k_0 - q) ,$$

where  $u_0$  is the sound wave energy and  $k_0$  is the sound wavenumber. Substituting this  $u(q)$  into the right hand side of the kinetic equation, we find that the first zero point term gives the attenuation of the sound wave:

$$\frac{du_0}{dt} = -u_0 \frac{QAc}{30} k_0^5 = \frac{\Lambda^2 h}{480 \rho_e \pi^2} k_0^5,$$

and the last two zero point terms give the spectrum of decay:

$$\frac{du_0}{dt} = u_0 QAc (k_0 - k)^2 k^3 / k_0 \quad \text{for } k < k_0 \text{ and zero for } k > k_0.$$

Landau and Lifshitz have suggested that zero point motion effects for macroscopic variables can be calculated by simply adding the zero point energy to thermal energy. However, this method would yield an infinite sound attenuation in contrast with the method presented in this chapter for which the results agree with quantum theoretical calculations.

The spontaneous decay of sound was measured for the first time in 1981<sup>20</sup> in an experiment in which high frequency sound was generated in doped calcium fluoride through the use of laser pulses to stimulate non-radiative transitions. The sound waves were detected through the fluorescence they induced in the stress-split energy levels of the  $E_u^{2+}$  doping ions. Phonon lifetimes were obtained from the time dependence of the fluorescence radiation and followed the  $k^5$  law.

APPENDIX A  
RAYLEIGH, LANGEVIN AND SOLITARY WAVE  
RADIATION PRESSURE IN A DISPERSIVE CONTINUUM

In this appendix we use dispersive nonlinear hydrodynamics to determine the motion and integrals of the motion for localized traveling disturbances in one dimensional continuous media.

Consider a disturbance which at first order can be expanded in a Fourier integral

$$\rho_1(x, t) = \int_{-\infty}^{\infty} dk \cdot \rho_1(k) \exp(ikx - i\omega t) \quad (A-1)$$

with

$$\omega(-k) = -\omega(k) \quad \text{and} \quad \rho_1(-k) = \rho_1^*(k)$$

so that the disturbance propagates in the "+x" direction. Similarly

$$\begin{aligned} J_1(x, t) &= \int_{-\infty}^{\infty} dk \cdot J_1(k) \exp(ikx - i\omega t) \\ &= \int_{-\infty}^{\infty} dk \frac{\omega(k)}{k} \rho_1(k) \exp(ikx - i\omega t) \end{aligned}$$

The inverse formula is:

$$\rho_1(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \cdot \rho_1(x, t) \exp(-ikx)$$



Substituting (A-1) into (1-36) yields for the second order contribution to the density

$$\frac{\partial^2 \rho_2}{\partial t^2} - c^2 \frac{\partial^2 \rho_2}{\partial x^2} + \alpha \frac{\partial^4 \rho_2}{\partial x^4} = \iint_{-\infty}^{\infty} dk dk' A(k, k') \exp(i(k+k')x - i(\omega+\omega')t) , \quad (A-2)$$

where

$$A(k, k') = -(k+k')^2 \frac{c^2 \Lambda}{\rho} \rho_1(k) \rho_1(k')$$

and  $\Lambda = 1 + \frac{\rho}{c} \frac{dc}{d\rho}$  .

The solution for  $\rho_2$  which vanishes at  $t=0$  and corresponds to a wave moving in the same direction as the original pulse is:

$$\rho_2 = \iint dk dk' \frac{A(k, k') \exp(i(k+k')x)}{-(\omega+\omega')^2 + c^2(k+k')^2 + \alpha(k+k')^4} \{ \exp(-i(\omega+\omega')t) - \exp(-i\omega(k+k')t) \}$$

which is made up of a particular plus homogeneous solution to (A-2):

$$\rho_2 = \rho_{2,p} + \rho_{2,h}$$

Similarly

$$J_2 = J_{2,p} + J_{2,h}$$

$$J_2 = \iint dk dk' \frac{A(k, k') \exp(i(k+k')x)}{-(\omega+\omega')^2 + c^2(k+k')^2 + \alpha(k+k')^4} \times \\ \left\{ \frac{\omega+\omega'}{k+k'} \exp(-i(\omega+\omega')t) - \frac{\omega(k+k')}{(k+k')} \exp(-i\omega(k+k')t) \right\}$$

where in each case the term in  $\exp[-i(\omega+\omega')t]$  is the particular solution.

The distinction between homogeneous and particular solution has a physical as well as mathematical importance. The frequency and wave number of the particular solution are not related by the dispersion relation (1-34). Therefore the particular solution is a property of the traveling first order wave. It is nonzero only in those regions where the first order disturbance is nonzero. The homogeneous solution however is an actual sound wave that propagates at the phase velocity  $\omega(k+k')/(k+k')$  and can therefore separate (for dispersive systems) from the original first order disturbance.

By choice of the initial condition

$$\int \rho_2 dx = 0 \quad (A-4)$$

but

$$\int J_2 dx = 2\pi\Delta \frac{c}{\rho_e} \int dk |\rho_1^2(k)| \quad (A-5)$$

$$\int \rho_{2,p} dx = 2\pi\Delta \frac{c^2}{\rho_e} \int dk |\rho_1^2(k)| + (v_g^2(k) - c^2) \quad (A-6)$$

$$\int J_{2,p} dx = 2\pi\Delta \frac{c^2}{\rho_e} \int dk |\rho_1^2(k)| v_g(k) + (v_g^2(k) - c^2) \quad (A-7)$$

where the group velocity  $v_g = d\omega/dk$ . Expanding the energy density (1-23) to second order:

$$\left[ U + \frac{1}{2} \rho v^2 \right]_2 = \mu_e \rho_2 + (c^2/2\rho_e) \rho_1^2 + \frac{1}{2} \rho_e v_1^2 + \frac{1}{2} \rho_e (\nabla \rho_1)^2 \quad (A-8)$$

leads to an energy due to the sound field:

$$\begin{aligned}
\int dx \left[ U + \frac{1}{2} \rho v^2 \right]_2 &= \mu_e \int dx \cdot \rho_2 + \int dx \cdot \rho_e v_1^2 \\
&= \mu_e \int dx \cdot \rho_2 + \frac{2\pi c^2}{\rho_e} \int dk |\rho_1^2| (1 + \alpha k^2 / c^2) . \quad (A-9)
\end{aligned}$$

From (A-4,5,9) we find to leading order (neglecting corrections of order  $\alpha k^2 / c^2$ ) that the ratio of total energy to total momentum of the sound field is

$$\frac{2c}{1 + \frac{\rho}{c} \frac{dc}{dp}} \quad (A-10)$$

which is the result first presented by Rayleigh.<sup>21</sup> Contrary to many references<sup>1,22</sup> the ratio is not  $c$  (except for a monatomic ideal gas which has the equation of state:  $PV^3 = \text{CONSTANT}$ ). The following comment should be made about our calculation of the energy and momentum of sound waves. We have considered only those effects which are second order in the field. It is entirely possible that the first order solutions  $\rho_1, J_1$  will have a net mass, energy and momentum. However, we have here focused upon the quadratic contributions since those are intrinsic properties of the acoustic field when the system is driven in an oscillatory fashion.

The momentum of the acoustic field (A-5) is proportional to  $\dot{A}$  and vanishes when the equation of state is such that

$$c = \text{constant} / \rho .$$

For a one dimensional lattice of masses on harmonic springs we have seen that  $c = \sqrt{\Phi/m}$  and  $a = \sqrt{\Phi m} / \rho$  which provides the proper explanation of

the statement that "phonons have no momentum." Normally this assertion is based upon the reasoning that in a crystal lattice as a wave passes each particle returns to its initial location and therefore the wave carries no momentum.<sup>23</sup> However, the fact that there is no drift in particle position merely requires  $\int J dt = 0$  which is quite different from the momentum of the wave which is given by  $\int J dx$  which is proportional to  $\Lambda$  and vanishes, only for the particular equation of state appropriate to the usually quantized harmonic lattice.

Next consider a first order disturbance which is peaked so that its spread in wave number is much less than the wave number " $k_0$ ". The homogeneous solution will separate from the disturbance<sup>3</sup> and its particular solution so that in time three pulses will appear: one with a mass and momentum given by (A-6,7) traveling at  $v_g$ , another with a mass given by the negative of (A-6) and a momentum given by the difference of (A-5) and (A-7) traveling at  $c$  and finally a pulse traveling at  $v_g(2k_0)$  which has zero mass and momentum to this order. These assertions follow directly from substituting a peaked distribution for  $\rho_1(k)$ .

Even when there is no net mass in the original disturbance, the dispersion separated pulse contains a net mass given by (A-6). Thus the total energy of this pulse involves the chemical potential (A-9). Neglecting corrections of order  $\alpha k^2/c^2$  in (A-9), we find that this pulse has the total energy:

$$\frac{2\pi c^2}{\rho_e} \int dk |\rho_1|^2 \left\{ 1 + \frac{\mu_e \Lambda}{v_g^2 - c^2} \right\} \quad (A-11)$$

An important distinction can be made as regards the two terms that contribute to the total energy. One of the terms (the first in the integral) should be regarded as a mechanical energy since dissipative processes can eventually convert this part into heat. The other term (involving  $\mu_e$ ) will always remain at this value regardless of the irreversible effects. Forming now the ratio of momentum to mechanical energy for the separated pulse we obtain

$$\frac{v_g(k_0)\Delta}{v_g(k_0)^2 - c^2} \quad (A-12)$$

We note that when  $v_g < c$ ,  $\rho_{2,p} < 0$  and if  $v_g > c$ ,  $\rho_{2,p} > 0$  and the pulse moves faster than  $c$ .

Bassett and Pryce<sup>3</sup> considered the quantum theory of one dimensional dispersive waves in an anharmonic lattice. Though they obtained Eq. (A-12) they did not mention the pulse traveling at  $v_q(2k_0)$  probably because in the quantum perturbation scheme employed by them the expectation values of energy, momentum vanished for that pulse. Furthermore, the contributions to the total energy proportional to  $\mu_e$  were not included in their work. This difference comes about because they considered the propagation of sound relative to an equilibrium lattice that is unstressed. For such a lattice  $\mu_e = 0$ ; as can be seen from the relation (VIZ: 18)

$$E = UV = M\mu - PV \quad ,$$

where  $E, V, M$ , are the total energy, volume, mass of the system in the mechanical equilibrium state ( $\vec{v}_0 = 0$ ). As is commonly done we take  $E = 0$  for the mechanical equilibrium state. Thus if  $P$  is also zero in this state we find that  $\mu$  must also vanish.

Had we included the underlined terms in Eq. (1-30) and so calculated corrections to the momentum (A-5) of order  $\alpha k_0^2/c^2$  we would have found

$$\int dx J_2 = \frac{\pi c}{\rho_e} \left\{ \Delta + \frac{\rho_e k_0^2}{2c^2} \left( \rho^{-4} \frac{d\Psi \rho^5}{d\rho} - \frac{3}{2} \Psi \Delta \right) \right\} \int dk |\rho_1^2(k)| \quad (A-13)$$

where of course  $(\rho_1^2(k))$  is peaked near  $k_0$ . The corrections proportional to  $k_0^2$  still vanish for the 1D harmonic lattice because in that case  $\Psi = \alpha/\rho_e a^5 \sim 1/\rho^5$ . In general the terms will not vanish.

The relation (A-12) for the ratio of momentum to energy has strong similarities to the properties of solitons<sup>24</sup> which are solutions to Eq. (1-36) in one dimension that have a stationary form:

$$\rho = \rho_e + \delta\rho(x-ct) \quad (A-14)$$

From the continuity law this requires the mass flow to be

$$J(x-ut) = u\delta\rho(x-ct)$$

so that Eq. (1-36) yields

$$(u^2 - c^2)\delta\rho + \alpha \frac{\partial^2 \delta\rho}{\partial x^2} = \frac{c^2}{\rho_e} \left\{ \frac{\rho_e}{c} \frac{dc}{d\rho} + \frac{u^2}{c^2} \right\} (\delta\rho)^2, \quad (A-15)$$

where terms cubic in  $\delta\rho$  have been dropped along with two space derivatives. This can be integrated directly to yield

$$\delta\rho = - \frac{3(c^2 - u^2)\rho_e}{2\Delta c^2} \operatorname{sech}^2 \left\{ \left[ \frac{c^2 - u^2}{4\alpha} \right]^{\frac{1}{2}} (x - ut) \right\} \quad (A-16)$$

where we have taken  $\alpha > 0$  so that  $u < c$ . Thus when the sound pulse travels at  $v_g > c$  the soliton moves slower than  $c$ , so that the mass contained in

the sound pulse has opposite sign to that of the soliton. From (A-16) we see that the small quantity for a soliton is  $c-u$ . If we set

$$\frac{c-u}{c} = \epsilon < 1 \quad (\text{A-17})$$

then

$$\delta\rho(0,0) = -\frac{3\rho_0}{\Lambda} \epsilon + \text{higher orders in } \epsilon$$

$$\frac{\partial^2 \delta\rho}{\partial x^2} = \frac{3\rho_0 c^2}{\alpha \Lambda} \epsilon^2 + \text{higher orders in } \epsilon \quad (\text{A-18})$$

so that differentiation of some property of a soliton increases its order of smallness. For a sound wave both amplitude and phase are independent but for the soliton the two are related by the requirement (A-14). In view of (A-17) we were justified in substituting  $c$  for  $u$  on the right hand side of (A-15) in order to arrive at (A-16) valid to this order of accuracy. Similarly to the order of  $\epsilon$  used here the underlined terms in Eq. (1-30) are higher order.

The momentum of the soliton is

$$\int dx J = u \int dx \delta\rho = \frac{u\Lambda}{u^2 - c^2} \frac{c^2}{\rho} \int dx (\delta\rho)^2 \quad (\text{A-19})$$

Introducing the mechanical energy as before we find that the ratio of momentum to mechanical energy for a soliton is

$$\frac{u\Lambda}{u^2 - c^2} \quad (\text{A-20})$$

With  $\alpha > 0$  a soliton exists with negative mass moving slower than  $c$ .

There is another wave packet which has kinematic properties very similar to this soliton (see Eq. A-12) which moves faster than  $c$  and however

spreads (disperses) very slowly in time. The inverse scattering method<sup>25</sup> makes very general statements about how initial disturbances evolve into solitons. If the initial pulse has the appropriate sign then at a sufficiently later time (neglecting dissipation) at least one soliton must appear. Thus one might wonder if the dispersion separated pulse will eventually turn into a soliton. The answer, however, is no for as we have seen the net mass in the acoustic pulse has opposite sign from the soliton appropriate to the given sign of  $\alpha$ .

Another quantity of interest in acoustics is the so-called Langevin radiation pressure<sup>26</sup> which is the net force exerted on a unit area of side wall as a wave runs down the tube:

$$F_L = \int dt (\delta P) ,$$

where to second order the pressure is

$$\delta P = c^2 \rho_2 + c \frac{\partial c}{\partial \rho} \rho_1^2 + \frac{1}{2} \frac{d\rho \bar{v}}{d\rho} (\nabla \rho_1)^2 .$$

Using (A-3) for  $\rho_2$  we find for  $F_L$

$$F_L = \pi \frac{c}{\rho_e} \left[ \frac{\rho}{c} \frac{dc}{d\rho} - 1 \right] \int dk \cdot \rho_1^2(k)$$

which is the usual result. The contribution to the Langevin pressure from the dispersion separated pulse is given to usual order of accuracy by:

$$F_{L,p} = c^2 \int dt \cdot \rho_{2,p} + c \frac{dc}{d\rho} \int dt \cdot \rho_1^2$$



$$= 2\pi \left\{ \frac{c^4}{\rho_e} \frac{\Lambda + \frac{k^2}{2c^2\rho^3} \frac{d\Psi}{d\rho}}{v_g(v_g^2 - c^2)} + c \frac{\partial c}{\partial \rho} \right\} \rho_1^2(k) dk. \quad (A-21)$$

To leading order in  $\omega k^2/c^2$  the term in  $\Lambda$  dominates the others. We have included the term in  $d\rho^5\Psi/d\rho$  which comes from the underlined term in Eq. (1-30) since it is of the same order in  $\omega k^2/c^2$  as the usual term  $cdc/d\rho$ . For a harmonic lattice the term in  $cdc/d\rho$  is the leading term. It must be noted that although this result becomes singular in the limit  $v_g \rightarrow c$  it also requires more and more time for the wave packet to separate in this limit.

A connection can be made between the nonlinear continuum coefficient  $\Lambda$  and the anharmonic terms in the one dimension lattice. Generalizing (1-3) to:

$$L = \sum_j \frac{m}{2} \dot{x}_j^2 - V, \quad (A-22)$$

where the potential energy is

$$V = \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} \frac{\Phi_n}{n!} (x_{i+1} - x_i - a_0)^n \quad (A-23)$$

where  $x_i$  = position of the particle which in mechanical equilibrium is located at " $ia_0$ " where  $a_0$  is the unstressed ( $P=0$ ) equilibrium separation. By definition the  $\Phi_n$  are independent of " $P$ ". If a steady external force " $P$ " acts on this system the equilibrium spacing will change to " $a$ " such that

$$P = - \sum_{n=2}^{\infty} \frac{\Phi_n}{(n-1)!} (a - a_0)^{n-1} \quad (A-24)$$

determines  $a(P)$ . The speed of sound for this one dimensional anharmonic lattice is

$$c^2 = \frac{\partial P}{\partial \rho} = - \frac{a^2}{m} \frac{\partial P}{\partial a} = \frac{a^2}{m} \sum_{n=2}^{\infty} \frac{\Phi_n}{(n-2)!} (a - a_0)^{n-2} \quad (A-25)$$

and

$$1 + \frac{\rho}{c} \frac{dc}{d\rho} = 1 - \frac{a}{c} \frac{dc}{da} = - \frac{a^3}{2mc^2} \sum_{n=3}^{\infty} \frac{\Phi_n}{(n-3)!} (a - a_0)^{n-3} \quad (A-26)$$

In the stressed lattice we can refer the  $x_{i+1} - x_i$  to the separation "a" by rewriting (A-23) as

$$V = \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} \frac{\Phi_n}{n!} (x_{i+1} - x_i - a + a - a_0)^n \quad (A-27)$$

and using the binomial expansion coefficients  $c_{mn}$  to write

$$V = \sum_{n=0}^{\infty} \frac{\Phi_n(a)}{n!} (x_{i+1} - x_i - a)^n \quad (A-28)$$

where

$$\frac{\Phi_n(a)}{n!} = \sum_{m=n}^{\infty} \frac{\Phi_m}{m!} c_{mn} (a - a_0)^{m-n} \quad (A-29)$$

yields the effective spring constants of the stressed lattice in terms of the spring constants of the unstressed lattice. In particular

$$\Phi_2(a) = 2 \sum_{m=2}^{\infty} \frac{\Phi_m}{m!} c_{m2} (a - a_0)^{m-2} \quad \text{and}$$

$$\Phi_3(a) = 6 \sum_{m=3}^{\infty} \frac{\Phi_m}{m!} c_{m3} (a - a_0)^{m-3} ,$$

and the term in  $\phi_0$  adds a constant to the potential whereas the term in  $\phi_1$  corresponds to a telescoping series which also contributes zero to the potential. Note also that:

$$\frac{\partial \phi_0(a)}{\partial a} = \phi_1(a) \quad , \quad \frac{\partial \phi_1(a)}{\partial a} = \phi_2(a) \quad , \quad \text{and} \quad \frac{\partial \phi_2(a)}{\partial a} = \phi_3(a) \quad . \quad (\text{A-30})$$

Since

$$\phi_1(a) = -P \quad ,$$

we find for the sound velocity

$$c^2 = \frac{a^2}{m} \phi_2(a) \quad (\text{A-31})$$

and for the Gruneisen coefficient

$$\Lambda = 1 + \frac{\rho}{c} \frac{dc}{d\rho} = - \frac{a\phi_3(a)}{2\phi_2(a)} \quad , \quad (\text{A-32})$$

which relates the hydrodynamic nonlinear parameter to the cubic anharmonic coefficient of the lattice for the one dimensional problem.

A precise transition from the Lagrangian (A-22) to the nonlinear hydrodynamics can be made by replacing

$$\sum_i \rightarrow \frac{1}{a} \int d\chi$$

$$\frac{\partial x}{\partial \chi} = \frac{\rho_e}{\rho}$$

$$x(\chi+a) = x(\chi) + a \frac{\partial x}{\partial \chi} + \dots \quad ,$$

where  $x(\chi, t)$  is the location of a particle at time  $t$  which was at point  $\chi$  at time "0", so that

$$v = \left( \frac{\partial x}{\partial t} \right)_{\chi}$$

## APPENDIX B

### THE EFFECT OF DECAY BANDWIDTH ON ATTENUATION

In the second chapter, we calculated the attenuation of sound due to nonlinear interaction with a background and showed that the magnitude of the attenuation was sensitive to the nature of the dispersion. In this appendix, we show that this sensitivity is absent when either the sound or background waves are attenuating significantly. We will also calculate the change in the wave speed that results from the nonlinear interactions.

A sound wave that is attenuating has an inherent uncertainty in frequency, which is called the decay bandwidth. If this uncertainty, which is dissipative dispersion, is much larger than the reversible dispersion then the net effect of dispersion on the attenuation is negligible.

To calculate these nonlinear effects, we use a perturbation method developed by Bedeaux and Mazur.<sup>27</sup> The method is to add to the fluid equations an external driving force and a fluctuating force. As an assumption, both forces drive the equations in an approximately linear manner. The purpose of the external force is to identify a Green's function, while the fluctuating force by design creates the background. The presence of the background changes the Green's function through nonlinear interactions. From this change, we directly calculate the attenuation and speed of a sound wave.

The effect of the decay bandwidths will be simulated by adding

to the fluid equations a dissipative stress tensor of a Navier-Stokes form. The attenuation of sound caused directly by this addition will be ignored but the resulting bandwidth which will appear in the non-linear attenuation will be replaced by the true bandwidth.

Before applying the perturbation method, we make some simplifying modifications and assumptions. First since the external and fluctuating forces are weak, we expand the fluid equations in the small deviations about the rest fluid, and keep terms which are at most bilinear in the deviations. Second we neglect the nonlinear terms in the fluid velocity. (In Chapter II, the velocity contributions were calculated by a different method.) Third we assume that the wavenumber of the sound wave is much less than the background wavenumbers. Finally, we will assume that the dissipative and reversible dispersions are small.

In Chapter I, we derived the dispersive fluid equations:

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \vec{\nabla} s = 0 \quad (\text{conservation of entropy}) \quad , \quad (\text{B-1})$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (\text{conservation of mass}) \quad , \quad \text{and} \quad (\text{B-2})$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{1}{\rho} \frac{\partial U}{\partial s} \vec{\nabla} s + \vec{\nabla} \left( - \frac{\partial U}{\partial \rho} + \frac{\partial}{\partial x_i} \frac{\partial U}{\partial \nabla_i \rho} \right) \quad (\text{Newton's law}) \quad . \quad (\text{B-3})$$

To expand these fluid equations in small deviations about the rest fluid, we let

$$\rho = \rho_e + \delta \rho(\vec{x}, t) \quad , \quad \vec{v} = \delta \vec{v}(\vec{x}, t) \quad , \quad \text{and} \quad s = s_e + \delta s(\vec{x}, t) \quad ,$$

where  $\rho_e$  and  $s_e$  are the density and specific entropy of the rest fluid and the quantities with the prefix  $\delta$  are the deviations from rest. We

then substitute these expressions into the fluid equations, Taylor expand the internal energy about the rest state, and keep only terms at most bilinear in the deviations. We begin by considering the entropy equation (B-1):

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \vec{\nabla} s = 0 \quad .$$

Substituting the deviation expressions, we find:

$$\frac{\partial}{\partial t} \delta s + \delta \vec{v} \cdot \vec{\nabla} \delta s = 0 \quad .$$

This equation states that the deviation in entropy is proportional to a small quantity times itself, which implies that the entropy deviation is approximately zero and entropy per mass is constant. Since entropy is constant, the expansion of the internal energy to second order in the density deviation is

$$U(\rho, s, \vec{v}) = U(\rho_e, s_e, 0) + \left. \frac{\partial U}{\partial \rho} \right|_e \delta \rho + \frac{1}{2} \left. \frac{\partial^2 U}{\partial \rho^2} \right|_e \delta \rho^2 + \frac{\partial U}{\partial (\vec{v}^2)} \frac{(\vec{v})^2}{2} \quad ,$$

and the two remaining fluid equations (B-2,3) in terms of the deviations are:

$$\frac{\partial \delta \rho}{\partial t} + \vec{\nabla} \cdot (\delta \rho \delta \vec{v}) + \rho_e \vec{\nabla} \cdot \delta \vec{v} = 0 \quad \text{and}$$

$$\frac{\partial \delta \vec{v}}{\partial t} + (\delta \vec{v} \cdot \vec{\nabla}) \delta \vec{v} = - \frac{c^2}{\rho} \vec{\nabla} \delta \rho - \frac{\epsilon}{2\rho} \vec{\nabla} (\delta \rho)^2 - \alpha \vec{\nabla} \vec{v}^2 \delta \rho \quad ,$$

where

$$c^2 = \rho_e \left. \frac{\partial^2 U}{\partial \rho^2} \right|_e \quad , \quad \epsilon = \rho_e \left. \frac{\partial^3 U}{\partial \rho^3} \right|_e \quad , \quad \text{and} \quad \alpha = \rho_e \left. \frac{\partial U}{\partial (\vec{v}^2)} \right|_e \quad .$$

If we take the time derivative of the first equation, subtract the gradient of the second equation, and neglect nonlinear terms in the

velocity, we have one equation for mass density:

$$\square \delta \rho = -\frac{1}{2}\epsilon \nabla^2 \delta \rho^2, \quad (\text{B-4})$$

where

$$\square \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + \alpha \nabla^4.$$

If we had included the Navier-Stokes dissipative tensor in the fluid equation pertaining to Newton's law (B-3), we would have one alteration:

$$\square \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + \alpha \nabla^4 + v_b \nabla^2 \frac{\partial}{\partial t},$$

where  $v_b$  is four-thirds the shear viscosity plus the bulk viscosity, which will be assumed to be a constant.

To apply the Green's function method, we add a driving force and a fluctuating force to the one remaining fluid equation (B-4), which is our basic equation of interest:

$$\square \delta \rho + \frac{\epsilon}{2} \nabla^2 \delta \rho^2 = T(\vec{x}, t) + F(\vec{x}, t) \quad (\text{B-5})$$

where  $T$  is the fluctuating force and  $F$  is the driving force. Next we Fourier analyze this equation:

$$(-\omega^2 + W(k)^2) \delta \rho - \frac{\epsilon}{2} k^2 \delta \rho^2 = T + F,$$

where  $\delta \rho$ ,  $\delta \rho^2$ ,  $T$ , and  $F$  are now Fourier transforms and  $W(k)^2 \equiv c^2 k^2 + \alpha k^4 + i v_b \omega k^2$ .

When the forces are weak, we can initially neglect the nonlinear term, and treat the density deviations created by each force as co-existing independently. In this approximation, we let  $\delta \rho = \delta \rho_F + \delta \rho_T$ , where  $\delta \rho_T$  and  $\delta \rho_F$  are created by the fluctuating and driving forces,

respectively. The deviations,  $\delta\rho_F$  and  $\delta\rho_T$ , satisfy the equations:

$$\delta\rho_F = G_0 F \quad \text{and} \quad \delta\rho_T = G_0 T, \quad (B-6)$$

where  $G_0$  is the linear Green's function:  $G_0^{-1} = (-\omega^2 + W(k)^2)$ .

The nonlinear term in the fluid equation (B-5) is the source of interaction between  $\delta\rho_T$  and  $\delta\rho_F$ . Including this term, we find a more accurate equation for  $\delta\rho_F$ :

$$(G_0^{-1} + L)\delta\rho_F = F,$$

where  $L$  is the following operator:

$$L = -\epsilon k^2 \delta\rho_T,$$

where it is implicit that  $L$  involves a convolution:

$$Lf = -\epsilon^2 k^2 \int dq \delta\rho_T(k-q) \cdot f(q).$$

We then formally invert the equation for  $\delta\rho_F$  and ensemble average over the fluctuating quantity  $\delta\rho_T$ :

$$\delta\rho_F = GF, \quad (B-7)$$

where  $G$ , which is identified as the Green's function in the presence of the background, is

$$G = \langle (1 + G_0 L)^{-1} \rangle G_0.$$

From the singularities of the Green's function, we will determine both the attenuation and speed of a sound wave. To do this we let the external driving force be

$$F(\vec{x}, t) = \exp(i\vec{k} \cdot \vec{r}) \delta(t),$$



and then use the Green's function and Eq. (B-7) to find the response

$\delta\rho_F$ :

$$\delta\rho_F = \exp(i\vec{k}\cdot\vec{r}) \int_{-\infty}^{+\infty} d\omega G(\omega, \vec{k}) \exp(i\omega t) .$$

Using contour integration, we identify the propagating modes for wave-number,  $\vec{k}$ , to be

$$\delta\rho_F = \exp(i\vec{k}\cdot\vec{r} + i\omega_j t) ,$$

where  $\omega_j$  are the simple poles of the Green's function for  $\text{Im}\omega > 0$ . The imaginary part of  $\omega_j$  is the attenuation coefficient and the real part divided by the wavenumber is the wave speed.

For our fluid equation, the Green's function will be of the form:

$$G = [ \omega^2 - W(k)^2 + g(\omega, \vec{k}) ]^{-1} ,$$

where the function  $g$  will be transcendental. To find the singularities of the Green's function, we will presume that the presence of function  $g$  does not significantly alter the positions of the poles so that the dependence of  $g$  on  $\omega$  can be removed by setting  $\omega$  equal to its approximate value in the vicinity of the pole:

$$G = [ \omega^2 - W(k)^2 + g(W(k), \vec{k}) ]^{-1} ,$$

The pole of the Green's function is at the location:

$$\omega_j = [ W(k) - g(W(k), \vec{k}) ]^{\frac{1}{2}} .$$

Expanding the square root in a Taylor series about  $W(k)$ , we find:

$$\omega_j = W(k) - \frac{1}{2} g(W(k), \vec{k})/W(k) .$$

The imaginary part of  $\omega_j$  is the attenuation coefficient,  $\beta$ :

$$\beta = \text{Im } g(W(k), \vec{k}) / 2W(k) , \quad (B-8)$$

and the real part divided by the wavenumber is the wave speed  $c(k)$ :

$$c(k) = \text{Re } \frac{1}{k} \left[ W(k) - \frac{1}{2} g(W(k), \vec{k}) / W(k) \right] . \quad (B-9)$$

To find the Green's function in the presence of the background  $(\delta\rho_T)$ , we must evaluate the ensemble average of the quantity:  $(1+G_0 L)^{-1}$ . We average this quantity by first expanding in powers of  $L$ , which is presumed small, and then approximating by keeping the first three terms. The terms are  $1$ ,  $G_0 L$ , and  $G_0 L G_0 L$ . The average of the first term is itself. For the second, if we impose translational invariance, then  $\langle \delta\rho_T \rangle = 0$  and since  $-G_0 L = G_0 \epsilon k^2 \delta\rho_T$ , we deduce that  $\langle -G_0 L \rangle = 0$ . For the third term, we write the term acting on the driving force with explicit convolutions:

$$G_0 L G_0 L F = \epsilon^2 k^2 G_0(\omega, k) \iiint_{-\infty}^{\infty} d\omega' d^3 k' d\omega'' d^3 k'' G_0(\omega'', \vec{k}'') k''^2 F(\omega', \vec{k}') \times \\ \delta\rho_T(\vec{k}'' - \vec{k}', \omega'' - \omega') \cdot \delta\rho_T(\vec{k} - \vec{k}'', \omega - \omega'') .$$

The ensemble dependence in this expression is in the quantity:

$$\delta\rho_T(\vec{k}'' - \vec{k}', \omega'' - \omega') \cdot \delta\rho_T(\vec{k} - \vec{k}'', \omega - \omega'') ,$$

whose ensemble average requires knowledge of the fluctuating force.

To proceed we must design the fluctuating force to create our desired background which we assume is isotropic with spectral intensity  $u(k)$ . To meet this demand, we let the fluctuating force satisfy the following correlation relation:

$$\langle T(\chi, \vec{q}) T(\chi', \vec{q}') \rangle = \frac{\rho_e v_b}{4\pi} q^2 u(q) \delta(\chi + \chi') \delta(\vec{q} + \vec{q}') .$$

One can then verify that for this correlation of the fluctuating forces the average energy density per unit wavenumber of the sound field is given by

$$\frac{c^2}{\rho} \iiint_{-\infty}^{\infty} dk' d\omega' d\omega \langle \delta\rho_T(\omega, \vec{k}) \delta\rho_T(\omega', \vec{k}') \rangle = u(k) ,$$

where (B-6) is used for  $\delta\rho_T$ . Using this correlation relation and the equation relating  $\delta\rho_T$  and  $T$ , we have the necessary ensemble average:

$$\langle \delta\rho_T(\chi, \vec{q}) \delta\rho_T(\chi', \vec{q}') \rangle = \frac{\rho_e v_b}{4\pi} q^2 u(q) |G_0(\chi, \vec{q})|^2 \delta(\chi + \chi') \delta(\vec{q} + \vec{q}') .$$

The presence of the delta functions is a consequence of imposing translational invariance for space and time.

We have shown that the Green's function in the presence of a background is

$$G = ( 1 + G_0 g(\omega, \vec{k}) ) G_0 ,$$

where

$$g(\omega, \vec{k}) = \langle L G_0 L \rangle .$$

Since the function  $g$  is small, we rewrite this equation in the approximate form:

$$G = ( G_0^{-1} + g(\omega, \vec{k}) )^{-1} .$$

When we neglect the viscosity term in  $G_0$ , we have shown that the attenuation coefficient ( $\beta$ ) and the wave speed are simply related to the function  $g$  by Eqs. (B-8,9).

What remains to be done is the evaluation of the function

$$gF = \epsilon^2 k^2 \int \int \int \int_{-\infty}^{\infty} d\omega' d^3 k' d\omega'' d^3 k'' G_0(\omega'', \vec{k}'') k''^2 F(\omega', \vec{k}') \times$$

$$\langle \delta \rho_T(\vec{k}'' - \vec{k}', \omega'' - \omega') \cdot \delta \rho_T(\vec{k} - \vec{k}'', W(k) - \omega'') \rangle.$$

Substituting the expression for the ensemble average of squared density deviation, making the change of variables  $\vec{K} = \vec{k} - \vec{k}''$  and  $\Gamma = W(k) - \omega''$ , and integrating the single primed integrations, we find:

$$g = \frac{v_b}{4\pi^2} \epsilon^2 \rho_e k^2 \int d^3 K u(K) (\vec{K} - \vec{k})^2 K^2 \int d\Gamma G_0(\Gamma, \vec{K}) G_0(-\Gamma, -\vec{K}) G_0(W(k) - \Gamma, \vec{k} - \vec{K}).$$

Next we use contour integration to perform the integral over  $\Gamma$ , which gives the result:

$$g = -\frac{1}{8\pi c^2} \epsilon^2 \rho_e k^2 \int d^3 K u(K) (\vec{K} - \vec{k})^2 \sum_{\pm} \left[ (W(k) \pm W(K) + \frac{1}{2} v_b (K^2 + (\vec{k} - \vec{K})^2))^2 - W(\vec{k} - \vec{K})^2 \right]^{-1}.$$

To integrate over  $\vec{K}$  space, we will use spherical coordinates whose z-axis is chosen along  $\vec{k}$ . The integration over azimuthal angle is trivial and has the value twice pi.

Before proceeding to perform the integration over the polar angle, we will approximate the denominators of the integrand. Since the background wavenumbers are assumed to be much greater than the wavenumber of the sound wave, we will keep only the leading order in the ratio  $k/K$ . The dispersion is assumed small so that we keep only the leading order in  $\alpha$  and  $v_b$ . With these approximations, the expression for  $g$  becomes

$$g = -\frac{1}{4c^2} \epsilon^2 \rho_e k^2 \int dK \frac{u(K)}{K} \int_{-\pi}^{\pi} d(\cos\theta) \times$$

$$\pm \left\{ \frac{K^2 + k^2 - 2Kk \cos\theta}{-4\alpha K^2 k \cos^2\theta + (2c^2 k + 4\alpha K^2 k + 2i v_b c K k) \cos\theta \pm (2c^2 k + \alpha K^2 k + 2i c v_b K^2)} \right\}.$$

Integrating over the angle and keeping leading orders in  $\alpha$  and  $v_b$ , we have the result:

$$g = -\frac{1}{8c^4} \epsilon^2 \rho_e k^2 \int dK \frac{u(K)}{K} \int_{\pm} \left\{ (K+k)^2 \ln \left( \cos\theta \pm \left( 1 - \frac{3\alpha}{2c^2} K^2 + \frac{i v_b K^2}{c k} \right) - 2Kk \cos\theta \right) \right\} \Big|_{-\pi}^{\pi}.$$

To find the attenuation coefficient and wave speed, we must separate function  $g$  into its real and imaginary parts. This can be accomplished with the following:

$$\ln(x+iy) = \frac{1}{2} \ln(x^2+y^2) - i \arctan(x/y) + i\pi/2.$$

Using this relation and evaluating the function  $g$  at the limits of the polar integration, we find:

$$g = -\frac{1}{2c^4} \epsilon^2 \rho_e k^2 \int dK u(K) \left\{ 2 + \frac{1}{2} \ln \left( 1 + \left( \frac{2\omega_s}{v_b K^2} \right)^2 \right) - \frac{1}{2} \ln \left( 1 + \left( \frac{3\alpha \omega_s}{2 v_b c^2} \right)^2 \right) - \right. \\ \left. i \left( \arctan \left( \frac{2\omega_s}{v_b K^2} \right) + \arctan \left( \frac{3\alpha \omega_s}{2 v_b c^2} \right) \right) \right\}$$

where  $\omega_s$  is the approximate frequency of the sound wave (i.e.,  $\omega_s = ck$ ).

Before finding the attenuation coefficient and wave speed, we make three revisions. First we remove the dependence of  $g$  on viscosity by replacing the quantity  $\frac{2}{\nu K^2}$ , which is the lifetime of a wave of wave-number  $K$  in the viscous fluid, by the true lifetime which we will label  $\tau(K)$ . Second, we include the effect of nonlinear terms in velocity by redefining  $\epsilon$  as follows:

$$\epsilon \equiv \frac{2c^2}{\rho_e} \Lambda$$

where as before

$$\Lambda = 1 + \frac{\rho}{c} \frac{dc}{d\rho}$$

Finally we assume that the logarithm and arctangent in the expression for  $g$  do not vary much over the range of wavenumbers for which the background energy density ( $u(k)$ ) is significant. This assumption allows us to take the logarithm and arctangent outside the integral and evaluate them at the wavenumber for which the background energy is peaked. Incorporating these revisions, we find:

$$g = \frac{2}{\rho_e} \Lambda^2 k^2 E \left\{ 2 + \frac{1}{2} \ln \left[ \frac{1 + (\omega_s \tau)^2}{1 + ((3\alpha \bar{K}^2 / 4c^2) \omega_s \tau)^2} \right] - i \left[ \arctan(\omega_s \tau) + \arctan((3\alpha \bar{K}^2 / 4c^2) \omega_s \tau) \right] \right\}$$

where  $\bar{K}$  is the wavenumber for  $u(k)$  is maximum,  $E$  is the total energy density (i.e.,  $E = \int dk u(K)$ ), and  $\tau = \tau(\bar{K})$ .

The relationships between the function  $g$  and the attenuation coefficient and wave speed was shown to be Eq. (B-8,9). Using these relationships and the calculated  $g$ , we find to leading order in  $\alpha$  and  $\nu_b$ :<sup>28</sup>

$$\beta = \frac{\Lambda^2}{\rho_e c^2} \omega_s E \left( \arctan(\omega_s \tau) + \arctan((3\alpha \bar{K}^2/4c^2)\omega_s \tau) \right) \quad \text{and}$$

$$c(\omega_s) = c_0 + \frac{\Lambda^2}{2\rho_e c_0} E \left\{ \ln \left( \frac{1 + (\omega_s \tau)^2}{1 + ((3\alpha \bar{K}^2/4c^2)\omega_s \tau)^2} \right) + 4 \right\}$$

where  $c_0$  is the wave speed in the absence of nonlinearities or thermal motion.

The assumptions and approximations that we have made in calculating the wave speed and attenuation are appropriate to describe sound waves in liquid helium at low temperatures. The background is thermal motion for which we know from statistical mechanics the total energy density:

$$E = \frac{4\pi^5 (k_B T)^4}{15h^3 c^3},$$

where  $T$  is temperature,  $k_B$  is Boltzmann's constant, and  $h$  is Planck's constant.

The attenuation and wave speed depend on the product of the sound frequency and the lifetime of the background waves. When this product is much smaller than one, the attenuation coefficient becomes proportional to the sound frequency squared:  $\beta = \frac{\Lambda^2 E}{\rho_e c^2} \tau \omega_s^2$ , and the change in wave speed due to nonlinear effects is frequency independent:

$\Delta c = \frac{2\Lambda^2 E}{\rho_e c}$ . When the product  $(\omega_s \tau)$  lies in the range  $1 \ll \omega_s \tau \ll \frac{c^2}{3\alpha \bar{K}^2}$ , then the attenuation coefficient is the same as Westervelt's result for

a nondispersive fluid, and the change in wave speed is

$$\Delta c = \frac{\Lambda^2}{2\rho_e c} E (4 + \ln(1 + (\omega_s \tau)^2)).$$

When the product  $\omega_s \tau$  is large so that  $\frac{3\alpha\bar{K}^2}{c^2} \omega_s \tau \gg 1$ , then the attenuation coefficient is either zero or twice Westervelt's result depending on whether the dispersive parameter  $\alpha$  is negative or positive, respectively. The change in wave speed for  $\omega_s \tau \frac{3\alpha\bar{K}^2}{c^2} \gg 1$  becomes again frequency independent:  $\Delta c = \frac{\Lambda^2 E}{2c\rho_e} \left( 4 + \ln \left( \frac{c^2}{3\alpha\bar{K}^2} \right)^2 \right)$ . For liquid helium at 0.0 bars and .3° Kelvin, the value of the quantity  $\frac{c^2}{3\alpha\bar{K}^2}$  is predicted by experiment<sup>5</sup> to be roughly one hundred if the dispersion relation is assumed to be of the form (1-34).

In liquid helium, neither the lifetime of the background waves nor the fine details of the dispersion have been determined. For our results, it is clear that no value of the lifetime completely accounts for the experimental results. The discrepancy may be due to our implicit assumption that the sound wave lifetime was longer than that of the background waves.



## APPENDIX C

### WIEN'S DISPLACEMENT LAW FOR A PONDERABLE CONTINUUM

In the late nineteenth century, Wien<sup>29</sup> derived the displacement law which states that the electromagnetic radiation in equilibrium has its energy distributed over wavelengths according to the form:

$$u(k) = k^3 g(k/T) ,$$

where  $u$  is the energy per unit wavenumber,  $T$  is temperature,  $k$  is wavenumber, and  $g$  is an undetermined function. For a fluid, we will prove that the thermal energy is distributed over wavelengths in the following way:

$$u(k) = ck^3 f(ck/T) . \quad (C-0)$$

where  $c$  is the speed of sound and  $f$  is an undetermined function. The appearance of the speed of sound is due to the fact that the sound speed is not constant, unlike the speed of light.

Our derivation of the displacement law for a fluid is outlined as follows. First we show following the method of Whitham<sup>30</sup> that for a sound wave the wave energy divided by the frequency is an adiabatic invariant. Next we consider a quasistatic change in volume and use the fact that the ratio of wave energy to frequency is constant for each thermal component to calculate the change in thermal distribution. From this change, we will show that the thermal distribution as a function of the volume and wavenumber has the form:

$$u(k) = ck^3 h(k^3V) ,$$

where  $v$  is the volume and  $h$  is an undetermined function. We then use thermodynamics and hydrodynamics to show that the quantity  $\frac{vT^3}{c^3}$  remains constant for a quasistatic change in volume. Using this fact and the functional dependence of the thermal distribution on volume, we deduce dependence of the thermal distribution on temperature is of the form (C-0).

That the ratio of the wave energy to frequency is an adiabatic invariant will be shown using the principle of least action. Action is defined as the integral of the Lagrangian density which is the difference between the kinetic and potential energy densities. To describe a sound wave, we will show that the Lagrangian can be approximated and written in terms of the independent variables relating to the amplitude and phase of the wave. Demanding that action be a minimum, we will show that resulting dynamical equations imply that the ratio of energy to frequency is an adiabatic invariant.

The Lagrangian density that is appropriate for a sound wave in a fluid is of the form:

$$L = \frac{1}{2} \rho \vec{v}^2 - U(\rho) \quad ,$$

where  $\rho$  is mass density,  $\vec{v}$  is the fluid velocity, and  $U$  is the internal energy. Since the sound wave is presumed to be of low amplitude, we expand the Lagrangian density in terms of the deviations and keep only leading order terms:

$$L = \frac{1}{2} \rho_e \delta \vec{v}^2 - U(\rho_e) - \left. \frac{\partial U}{\partial \rho} \right|_e \delta \rho - \frac{c^2}{2\rho_e} \delta \rho^2 \quad ,$$

where  $\rho_e$  is the rest density,  $\delta\vec{v}$  and  $\delta\rho$  are the deviations, and  $c^2 \equiv \rho_e \frac{\partial^2 u}{\partial \rho^2} \Big|_e$ . The leading term in the expansion of the internal energy is neglected because it is constant and will have no effect when demanding that action be minimum. The second term is also neglected because its contribution to the sound wave energy is zero. The effective Lagrangian density is therefore:

$$L = \frac{1}{2} \rho_e \delta\vec{v}^2 - \frac{c^2}{2\rho_e} \delta\rho^2 .$$

Next we rewrite the Lagrangian in terms of the Lagrange coordinates using the relations:

$$\vec{v} = \frac{\partial \vec{\eta}}{\partial t} \quad \text{and} \quad \delta\rho = \rho_e \vec{\nabla} \cdot \vec{\eta} ,$$

where  $\eta$  is the displacement of the fluid particle from equilibrium. For a sound wave, the displacement can be written in terms of the amplitude and phase:

$$\vec{\eta} = \vec{A} \exp(i\theta) + \text{c.c.} ,$$

where  $\vec{A}$  is the amplitude and  $\theta$  is the phase to which frequency and wavenumber are related as follows:

$$\omega = - \frac{\partial \theta}{\partial t} \quad \text{and} \quad \vec{k} = \vec{\nabla} \theta .$$

When the density of the fluid is changing slowly, then the quantities  $\vec{A}$ ,  $\omega$ , and  $\vec{k}$  are also slowly varying. Substituting the form of the displacement into the Lagrangian density and neglecting the slow variations, we find

$$L = - \frac{\rho_e}{2} \text{Re}[\vec{A} \exp(i\theta) - \vec{A} \exp(-i\theta)]^2 \left\{ \left( \frac{\partial \theta}{\partial t} \right)^2 + c^2 (\vec{\nabla} \theta)^2 \right\} .$$

Since we are interested in global properties of the sound wave, we average the Lagrangian over a wavelength:

$$\langle L \rangle = \rho_e |\vec{A}|^2 \left[ \left( \frac{\partial \theta}{\partial t} \right)^2 + c^2 (\vec{\nabla} \theta)^2 \right]$$

Performing the variations of the independent variables  $\vec{A}$  and  $\theta$ , and demanding that action be a minimum, we find the equations:

$$\left( \frac{\partial \theta}{\partial t} \right)^2 = c^2 (\vec{\nabla} \theta)^2 \quad \text{and} \quad \frac{\partial}{\partial t} \left[ \rho_e |\vec{A}|^2 \frac{\partial \theta}{\partial t} \right] + \vec{\nabla} \cdot \left[ \rho_e |\vec{A}|^2 c^2 \vec{\nabla} \theta \right] = 0$$

Using vector analysis and algebra, we can use the two equations above to derive the following equation:

$$\frac{\partial}{\partial t} Q + \vec{\nabla} \cdot \left[ \frac{\vec{\nabla} \theta}{|\vec{\nabla} \theta|} c Q \right] = 0 \quad \text{where} \quad Q = \rho_e |\vec{A}|^2 \left\{ \left( \frac{\partial \theta}{\partial t} \right)^2 + c^2 (\vec{\nabla} \theta)^2 \right\} + \left( \frac{\partial \theta}{\partial t} \right)$$

This equation is in the form of a conservation equation for which the invariant quantity is  $Q$ . The excess energy density for the fluid is approximately:

$$E = \frac{1}{2} \rho_e \delta \vec{v}^2 + \frac{c^2}{2 \rho_e} \delta \rho^2$$

which if put in terms of  $\vec{A}$  and  $\theta$ , and averaged gives the energy of the sound wave:

$$E = \frac{\partial \theta}{\partial t} Q$$

Since the quantity  $\frac{\partial \theta}{\partial t}$  is the frequency and the quantity  $Q$  is invariant, we deduce that the energy divided by frequency is an adiabatic invariant. For the particular case of the quasistatic change in volume we have

$$\frac{d}{dV} (E/\omega) = 0 \quad (C-1)$$

We now consider a small quasistatic change in volume of a fluid at some initial temperature. The energy of thermal motion can be

viewed as being distributed in an unique way over the wavenumbers of the thermal waves. The quasistatic change in volume alters the distribution. If nonlinear interaction between waves is neglectable, then the change in the distribution can be determined by applying (C-1) to each of the waves that compose the thermal distribution. From the change in distribution we will derive information about the form of the distribution.

The proof that nonlinearities are neglectable is as follows. We imagine that the fluid is compressed with nonlinearities turned off so that the distribution changes according to (C-1). Now we turn on the nonlinearities. If we started from an equilibrium state and if the compression is reversible then there can be no entropy change due to nonlinearities and the new distribution generated from (C-1) must be an equilibrium distribution for which nonlinearities have no effect.

To find the change in distribution due to the quasistatic volume change, we let  $h$  be the mechanical function [ $h=h(k,V)$ ] such that the energy density in the wavenumber interval  $dk$  at  $k$  is  $hck^3dk$  (i.e. we start with a certain distribution  $h$  over  $k$  for the given volume and now change volume and therefore take  $h=h(k,V)$ ). Then the fact that the energy in  $dk$  divided by its frequency ( $ck$ ) is invariant implies the relation:

$$Vhk^2 dk = V'h'k'^2 dk' \quad , \quad (C-2)$$

where  $V$  is the volume and the primes denote the changed quantities. This relation can be simplified by expressing the primed wavenumber in terms of the change in volume. Boundary conditions at the walls containing the fluid demand that the wavenumber be inversely proportional

to the linear dimension of the system and since volume is proportional to the third power of the linear dimension, we can use simple calculus to derive the following relation which is valid to first order in the change in volume:

$$k' = \left[ 1 - \frac{\Delta V}{3V} \right] k, \quad (C-3)$$

where  $\Delta V$  is the change in volume. Substituting the  $k'$  from (C-3) into (C-2) and keeping terms to order  $\Delta V$ , we find  $h=h'$ . (C-4)

Using Eq. (C-4), we will show that the thermal distribution is of the form:

$$u(k) = ck^3 h(k^3 V). \quad (C-5)$$

First we expand the primed  $h$  in a Taylor series about the initial wavenumber and volume, and keep only terms first order in  $\Delta V$ :

$$h' = h(k', V') = h(k, V) + \frac{\partial h}{\partial k} \Delta k + \frac{\partial h}{\partial V} \Delta V. \quad (C-6)$$

Using the Eqs. (C-3), (C-4), and (C-6), we can relate the partial derivatives of  $h$ :  $\frac{k}{3} \frac{\partial h}{\partial k} = V \frac{\partial h}{\partial V}$ . The solution to this partial differential equation is  $h=h(k^3 V)$ , which proves the thermal distribution is of the form (C-5).

Our ultimate goal is to derive the displacement law, which will follow from (C-5) and the fact that the quantity  $VT^3/c^3$  is a thermodynamic adiabatic invariant. The proof that  $VT^3/c^3$  is invariant will proceed as follows. First we separate both the pressure and energy of the fluid into two pieces: one piece due to thermal motion and the other being that which exists in the absence of thermal motion (at zero degrees). Then we use (C-1) to derive the relationship between the pressure and energy of the excitations. With this relationship

and a form of the second law of thermodynamics we will show that the total energy of thermal motion has the following dependence on  $T$  and  $c$ :

$$U_x = \text{constant} \cdot \frac{T^4}{c^3} \quad (C-7)$$

Using this last relation and the first law of thermodynamics, we will find that  $VT^3/c^3$  is constant for a quasistatic change in volume.

For the fluid we write the pressure and total energy density in the sums:  $P=P_0+P_x$  and  $U=U_0+U_x$ , where the subscript "zero" refers to the values of the quantities at zero temperature and the subscript "x" refers to contribution from thermal motion. The energy density and pressure are related by the first law of thermodynamics:

$$P = - \left. \frac{\partial UV}{\partial V} \right|_s \quad (C-8)$$

At zero degrees, this equation reads as

$$P_0 = - \left. \frac{\partial U_0 V}{\partial V} \right|_s \quad (C-9)$$

Using (C-8) and (C-9), we can derive the equation that relates  $P_x$  and  $U_x$ :

$$P_x = - \left. \frac{\partial U_x V}{\partial V} \right|_s \quad (C-10)$$

A simple relationship between  $P_x$  and  $U_x$  can be derived from (C-1), (C-10), and the assumption that the speed of sound depends only on density. For a single wave, (C-1) implies that in a quasistatic change in volume the change in wave energy in terms of the change in frequency is given by the equation:

$$\frac{\Delta E}{E} = \frac{\Delta \omega}{\omega}$$

Since  $\omega=ck$ , the change in energy can be expressed in terms of  $\Delta k$  and  $\Delta c$ :

$$\frac{\Delta E}{E} = \frac{\Delta c}{c} + \frac{\Delta k}{k} \quad (C-11)$$

If  $c$  is only a function of density, then the change in  $c$  in terms of  $\Delta V$  is:

$$\Delta c = - \rho \frac{\partial c}{\partial \rho} \frac{\Delta V}{V} \quad (C-12)$$

Using (C-3), (C-11) and (C-12) we can express  $\Delta E$  in terms of  $\Delta V$ :

$$\Delta E = - \left[ \frac{1}{3} + \frac{\rho}{c} \frac{dc}{d\rho} \right] \frac{E}{V} \Delta V$$

Since this equation is true for each wave that composes the thermal distribution, it is also true for the whole distribution so that

$$\Delta(U_x V) = - \left( \frac{1}{3} + \frac{\rho}{c} \frac{dc}{d\rho} \right) U_x \Delta V$$

Comparing this equation to (C-10), we deduce that  $P_x$  is

$$P_x = \left[ \frac{1}{3} + \frac{\rho}{c} \frac{dc}{d\rho} \right] U_x \quad (C-13)$$

As the relation for  $\Delta E$  also applied only to adiabatic transformations, the dependence of  $U_x$  on  $T$  can be found using (C-13) and the second law of thermodynamics. A consequence of the second law, which arises from equating mixed partial derivatives of entropy, is the equation:<sup>31</sup>

$$\left. \frac{\partial U V}{\partial V} \right|_T = T \left. \frac{\partial P}{\partial T} \right|_V - P \quad (C-14)$$

Since  $\frac{\partial U_0 V}{\partial V} = P_0$ , we can write (C-14) for just  $P_x$  and  $U_x$ :

$$\left. \frac{\partial U_x V}{\partial V} \right|_T = T \left. \frac{\partial P_x}{\partial T} \right|_V - P_x$$

Substituting the  $P_x$  from (C-13), we have:



$$\left. \frac{\partial U_x v}{\partial v} \right|_T = \left[ \frac{1}{3} + \frac{\rho}{c} \frac{dc}{d\rho} \right] \cdot \left[ T \left. \frac{\partial U_x}{\partial T} \right|_v - U_x \right]$$

If we let  $Z = v/c^3$ , we can transform the last equation:

$$\frac{\partial}{\partial \ln z} \frac{v U_x}{T} = \frac{\partial}{\partial \ln T^3} \frac{v U_x}{T}$$

The general solution to this equation is

$$U_x = \frac{T}{v} F(\ln(z T^3)) ,$$

where  $F$  is as yet undetermined. Since  $U_x$  is intensive, the function  $F$  must be exponential and therefore the total energy density is given by

$$U_x = \text{constant } T^4/c^3 . \quad (C-15)$$

Using (C-10) and (C-15), we can prove that the quantity  $v T^3/c^3$  is invariant for a quasistatic change in volume. Substituting the  $P_x$  from (C-13) and the  $U_x$  from (C-15) into (C-10), we find with some algebra:

$$\left. \frac{\partial T}{\partial v} \right|_s = - \left[ \frac{1}{3} + \frac{\rho}{c} \frac{dc}{d\rho} \right] \frac{T}{v} . \quad (C-16)$$

The identity

$$\frac{1}{3} + \frac{\rho}{c} \frac{dc}{d\rho} = \frac{c^3}{3} \frac{\partial}{\partial v} \frac{v}{c^3}$$

is useful for the derivation. Using equation (C-16), we can show directly that

$$\left. \frac{\partial}{\partial v} \right|_s \left[ \frac{v T^3}{c^3} \right] = 0 . \quad (C-17)$$

An implication that can be made from (C-17) is that during any quasistatic change in volume, the volume is proportional to a constant times  $T^3/c^3$ . Substituting this form of  $v$  into (C-5), we have the

displacement law for a fluid:

$$u(k) = ck^3 f(ck/T) .$$

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